Poznámky z přednášek
Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

Lambda-kalkulus a funkcionální programování I
BONUS

Peter Černo, 2010
petercerno@gmail.com

Garant: prof. RNDr. Petr Štěpánek, DrSc.
E-mail: Petr.Stepanek@mff.cuni.cz
Domácí stránka: `http://ktiml.mff.cuni.cz/~stepanek/`

Anotace: Kombinatorické kalkuly a lambda kalkuly, netypované kalkuly, representovatelnost rekursivních funkcí. Churchova a Rosserova vlastnost a konsistence lambda kalkulu. Typovaný lambda kalkulus a jeho vztah k funkcionálnímu programovaní.

Sylabus:

1. Úvod. Kombinatorický kalkul Schoenfinkelův. Kombinatorické objekty jako representanti funkcí jedné proměnné. Representace funkcí více proměnných. Abstrakce vzhledem k proměnným a neurčitým. Kombinatorická úplnost kalkulu. Bezespornost kalkulu. Church-Rosserova vlastnost.

2. Kombinatorické kalkuly. Syntax Churchova kombinatorického kalkulu GammaK, neomezená abstrakce, syntaktický popis Schoenfinkelova kombinatorického kalkulu GammaI, omezená abstrakce. Gamma objekty v normálním tvaru.

3. Lambda kalkulus, neformální popis. Proměnné, volné a vázané, abstrakce omezená a neomezená. Lambda objekty v normálním tvaru. Church-Rosserova vlastnost a jednoznačnost lambda-objektů v normálním tvaru.

4. Lambda kalkuly. Syntaktický popis lambda objektů, volné a vázané proměnné, alfa a beta redukce. Další rozšíření lambda kalkulu, extensionalita, pravidlo delta.

5. Bezespornost kombinatorických kalkulů a lambda kalkulů. Church-Rosserova vlastnost a bezespornost kalkulů. Objekty, které nemají normální tvar. Paradoxy. Kombinatorická logika a predikátová logika: Curryho paradox.

6. Funkcionální programování a Lambda Kalkulus. Výpočetní model funkcionálního jazyka, redukce, normální tvary a funkcionální programování, řízení výpočtu ve funkcionálním programování.

7. Lambda kalkul ještě jednou. Konverse, aplikace a abstrakce. Lambda termy, kombinátory. Věty o pevném bodě.

8. Representace vyčíslitelných funkcí. Numerály, booleovské konstanty, uspořádané dvojice. Representovatelnost (totálních) rekursivních funkcí pomocí lambda termů.

9. Typový lambda kalkul. Rozšíření jazyka lambda kalkulu o typy objektů. Motivace: částečná správnost programů, efektivnost výpočtů. Typové proměnné a přiřazení hodnot typovým proměnným. Pravidla týkající se typů. Hierarchie teorií typového lambda kalkulu.

Cíl předmětu: Naučit teorii lambda-kalkulu, kombinatorické logiky a ukazat souvislosti s funkcionálním programováním.

Literatura:

1. Hendrik Pieter Barendregt Introduction to Lambda Calculus
`ftp://ftp.cs.ru.nl/pub/CompMath.Found/lambda.pdf`

This page is intentionally left blank.

02.10.2008 Th
PETER ČERNO

NAIL078 LAMBDA KALKULUS I

VYTLAČIŤ SLIDY !

λ-KALKUL
PRACUJE S TERMAMI

$$R(E) \xrightarrow[B(\alpha)]{} R'(E')$$

$$(\lambda x.x)(y) \xrightarrow[B]{} y$$

$$(\lambda x.xy)(z) \xrightarrow[B]{} zy$$

METÓDA DEKLARATÍVNEHO PROGRAMOVANIA

LAMBDA KALKULUS JE ZÁKLADOM
FUNKCIONÁLNEHO PROGRAMOVANIA (HASKELL)

π-KALKUL – DISTRIBUOVANÉ SYSTÉMY, ..

LP (LOG. PROG.) PRACUJE SA S FORMULAMI

$$\begin{cases} A_1 \leftarrow B_1^1, \dots, B_{n_1}^1 \\ \dots \\ A_j \leftarrow B_1^j, \dots, B_{n_j}^j \end{cases}$$

?

Q

PRE FORMULE PLATÍ VETA O EKVIVALENCII
EXTENSIONALITY
$\vdash C \leftrightarrow C'$, C' VZNIKNE Z C NAHRADENÍM
NEJAKÝCH VÝSKYTOV FORMULÍ B_i FORMULAMI D_i,
PRIČOM $\vdash B_i \leftrightarrow D_i$.

V LP JE PROGRAM FORMULA

V λ-KALKULU JE PROGRAM λ-TERM.

λ-TERM t NÁM DÁVA REFERENCIU

$t(E) \to t'(E')$

EKVIVALENCIA NIE JE ROZHODNUTEĽNÁ

$$\underbrace{(\lambda x.\, xy)}_{\substack{\lambda\text{-TERM} \\ \text{PROGRAM}}}(z) \to zy$$

PROGRAM VSTUP

REFERENČNE TRANSPARENTNÝ SYSTÉM

REFERENČNE OPALESCENTNÝ SYSTÉM

$\Gamma \vdash t : \sigma$

BÁZA DEKLARÁCIÍ TYPOV PREMENNÝCH

$x_0 : \tau_0, \ldots, x_n : \tau_n$

NETYPOVANÝ λ-KALKULUS

OPERÁCIA APLIKÁCIE: (FA)

OPERÁCIA ABSTRAKCIE: $M[x]$

$\lambda x.\, M[x]$ JE MAPOVANIE $x \mapsto M[x]$

PRAVIDLO REDUKCIE β

(β) $\underbrace{(\lambda x.\, M[x])N}_{\text{REDEX}} = \underbrace{M[x := N]}_{\text{CONTRACTUM}}$

02.10.2008 Th
PETER ČERNO

NAIL078 LAMBDA KALKULUS I

$$(\lambda y.xy)(\lambda x.x)[x:=N] = (\lambda y.Ny)(\lambda x.x)$$

NAHRADZUJÚ SA IBA VOĽNÉ PREMENNÉ

$FV(x) = \{x\}$ $\quad FV(MN) = F(M) \cup F(N)$

$FV(\lambda x.P) = FV(P) - \{x\}$

UZAVRETÝ λ-TERM: $FV(M) = \emptyset$

FUNKCIE VIAC PREMENNÝCH

MÁME $f(x,y)$

$F_x = \lambda y.\, f(x,y)$

$F = \lambda x.\, F_x$

$(Fx)y = F_x\, y = f(x,y)$

$F\, M_1 \dots M_n \overset{def.}{=} ((F\, M_1) M_2 \dots) M_n)$

$\lambda x_1 \dots x_n . f(x_1, \dots, x_n) \overset{def.}{=}$

$\lambda x_1 . (\lambda x_2 \dots (\lambda x_n\, f(x_1, \dots, x_n)) \dots)$

$F = \lambda xy.\, f(x,y)$

$Fxy = (\lambda xy.\, f(x,y))xy =$

$= (((\lambda x.(\lambda y.\, f(x,y)))x)y$

FORMÁLNY POPIS

2 NEKONEČNÉ MNOŽINY :
MNOŽINA KONŠTANT C, MN. PREMENNÝCH V

MNOŽINA λ-TERMOV Λ

$c \in C \Rightarrow c \in \Lambda$
$x \in V \Rightarrow x \in \Lambda$

$M, N \in \Lambda \Rightarrow (MN) \in \Lambda$
$M \in \Lambda, x \in V \Rightarrow (\lambda x. M) \in \Lambda$

ABSTRAKCIA: $\Lambda = C \mid V \mid \Lambda\Lambda \mid \lambda V \Lambda$

(β) $(\lambda x. M)N = M[x := N]$ IBA VOLNÉ VÝSKYTY x

POZNÁMKA $(\lambda x\, M) = (\lambda y\, M_x[y])$ JE (α)

PRAVIDLÁ =

(RULE - ξ) $M = M' \Rightarrow (\lambda x. M) = (\lambda x. M')$

09.10.2008 Th
PETER ČERNO

NAIL078 LAMBDA-KALKULUS I

FORMAL DESCRIPTION

INFINITE SET OF CONSTANTS C, VARIABLES V
$\longrightarrow$ SET OF LAMBDA-TERMS Λ

$c \in C \Rightarrow c \in \Lambda$, $v \in V \Rightarrow v \in \Lambda$
$M, N \in \Lambda \Rightarrow (MN) \in \Lambda$ (APPLICATION)
$M \in \Lambda,\ x \in V \Rightarrow (\lambda x.M) \in \Lambda$ (ABSTRACTION)

$$\boxed{\Lambda = C \mid V \mid (\Lambda\Lambda) \mid (\lambda V.\Lambda)}$$

FREE VARIABLES FV
$FV(x) = \{x\}$, $FV(c) = \emptyset$
$FV(MN) = FV(M) \cup FV(N)$
$FV(\lambda x.M) = FV(M) \setminus \{x\}$

CLOSED λ-TERM M: $FV(M) = \emptyset$

PRINCIPAL AXIOM SCHEME

$$\boxed{(\beta) \quad (\lambda x.M)N = M[x := N]}$$

AXIOMS FOR $=$

$\lambda \vdash M = N \equiv M$ AND N ARE β-CONVERTIBLE
$M \equiv N$ (ONLY RENAMING VARIABLES)
(α) $(\lambda x.M) = (\lambda y.M)[x := y]$ $\quad y \notin M$

$\lambda x_1 \dots x_n . M \sim (\lambda x_1.(\lambda x_2. \dots .(\lambda x_n . M)\dots))$
$M N_1 \dots N_n \sim ((\dots((MN_1)N_2)\dots)N_n)$!

$$S = \lambda xyz.\, xz(yz) =$$
$$= (\lambda x.(\lambda y.(\lambda z.((xz)(yz)))))$$
$$SMNL = (((SM)N)L) =$$
$$= (((\lambda y.(\lambda z.((xz)(yz))))[x:=M]\ N)L) =$$
$$= (((\lambda y.(\lambda z.((Mz)(yz))))\ N)\ L) =$$
$$= ((\lambda z.((Mz)(yz)))[y:=N]\ L) =$$
$$= ((\lambda z.((Mz)(Nz)))\ L) =$$
$$= ((Mz)(Nz))[z:=L] =$$
$$= \underline{\underline{((ML)(NL))}}$$

FIXED POINT THEOREM

(i) $\forall F \in \Lambda\ \exists X \in \Lambda : \quad \lambda \vdash (FX) = X$

$$\boxed{W := (\lambda x.(F(xx))) \qquad X := (WW)}$$

$$X \equiv (WW) \equiv ((\lambda x.(F(xx)))W) \overset{(\beta)}{=}$$
$$= (F(xx))[x:=W] = (F(WW)) \equiv (FX)$$

(ii) FIXED POINT <u>COMBINATOR</u>

$$Y \equiv (\lambda f.((\lambda x.(f(xx)))(\lambda x.(f(xx))))))$$

$$\forall F \in \Lambda : \quad \lambda \vdash (F(YF)) = (YF)$$

FROM (i): $(YF) = (WW)$ ...

09.10.2008 Th
PETER ČERNO

NAIL078 LAMBDA-KALKULUS I

CONTEXT LEMMA

GIVEN A CONTEXT $C[f,x]$

$\boxed{\exists F \in \Lambda}\ \forall X \in \Lambda : \lambda \vdash (FX) = C[F,X]$

$\forall X \in \Lambda \quad \lambda \vdash (FX) = C[F,X] \Leftarrow$

$\Leftarrow \lambda \vdash (Fx) = C[F,x] \Leftarrow$

$(Fx) = C[F,x] \Rightarrow (\lambda x.(Fx)) = (\lambda x.C[F,x])$

$\Rightarrow ((\lambda x.(Fx))X) = ((\lambda x.C[F,x])X)$ $\quad$ ξ-RULE

$\Rightarrow (Fx)[x:=X] = C[F,x][x:=X]$

$(FX) = C[F,X]$

$\Leftarrow \lambda \vdash F = (\lambda x.C[F,x]) \Leftarrow$

$\Leftarrow \lambda \vdash F = ((\lambda f.(\lambda x.C[f,x]))F) \Leftarrow$

$\Leftarrow \lambda \vdash F = (Y\underbrace{(\lambda f.(\lambda x.C[f,x]))}_{T})$

$F = (YT)$

$\Rightarrow (YT) = (T(YT))$

$\Rightarrow F = (TF)$

! KDE $\underline{\underline{Y = (\lambda f.((\lambda x.(f(xx)))(\lambda x.(f(xx)))))}}$

STAČÍ POLOŽIT:

$\underline{\underline{F := ((\lambda f.((\lambda x.(f(xx)))(\lambda x.(f(xx))))))(\lambda f.(\lambda x.C[f,x]))}}$

CHURCH NUMERALS $\quad c_n = \lambda f x . f^n(x)$

$$A_+ \equiv \lambda xypq . xp(ypq)$$

$$A_+ c_n c_m \equiv \big((\lambda xypq . xp(ypq))(\lambda fx . f^n(x))\big)(\lambda fx . f^m(x))$$

$$= (\lambda ypq . (\lambda fx . f^n(x))p(ypq))(\lambda fx . f^m(x)) =$$

$$= \lambda pq . (\lambda fx . f^n(x))p((\lambda fx . f^m(x))pq) =$$

$$= \lambda pq . (\lambda x . p^n(x))((\lambda x . p^m(x))q) =$$

$$= \lambda pq . (\lambda x . p^n(x))(p^m(q)) =$$

$$= \lambda pq . p^n(p^m(q)) =$$

$$= \lambda fx . f^{n+m}(x) = \underline{\underline{c_{n+m}}}$$

$$A_+ c_n c_m = \lambda pq . (c_n p(c_m pq)) =$$

$$= \lambda pq . (\lambda x p^n(x)(p^m(q))) = \lambda pq . p^n(p^m(q))$$

$$A_\times c_n c_m = (\lambda xyz . x(yz)) c_n c_m =$$

$$= \lambda z . c_n(c_m z) = \lambda z . c_n(\lambda x . z^m(x)) =$$

$$= \lambda z . ((\lambda fx . f^n(x))(\lambda x . z^m(x))) =$$

$$= \lambda z . (\lambda x . ((\lambda x . z^m(x))^n(x))) =$$

$$= \lambda zx . ((\lambda x . z^m(x))^n(x)) = \lambda zx . z^{m \times n}(x) = \underline{\underline{c_{m \times n}}}$$

$$A_{exp} c_n c_m = (\lambda xy . yx) c_n c_m = c_m c_n =$$

$$\cancel{= (\lambda fx . f^m(x)) c_n = \lambda x . (c_n)^m(x) = \lambda z . (c_n)^m(z) =}$$

$$\cancel{= \lambda z_1 . (\lambda fx . f^n(x))^m(z_1) = \lambda z_1 z_2 . \underbrace{((\lambda fx . f^n(x))(z_m))^n(z_4) \ldots}^n z_2 z_1}$$

$$\cancel{(\lambda fx . f^n(x))^m = (\lambda fx . f^n(x))(\lambda fx . f^n(x))^{m-1} =}$$

$$\cancel{= \lambda z . ((\lambda fx . f^n(x))^{m-1})^n(z)}$$

$$\cancel{\lambda z_1 z_2 . (\lambda z_2 x . z_2^n(x))}$$

$$z_2 z_1 . \; ?$$

11.10.2008 Sa
PETER ČERNO

NAIL078 LAMBDA-KALKULUS I

$$A_* = \lambda xyz.\, x(yz)$$

$$A_*\,(A_{exp}\, c_n\, c_{m-1})\, c_n =$$
$$= \lambda z.\ (c_{m-1} c_n)(c_n z) =$$
$$= \lambda z.\ ((\lambda fx.\, f^{m-1}(x))\, c_n)\,(\lambda y.\, z^n(y)) =$$
$$= \lambda z.\ ((\lambda x.\, (\lambda zy.\, z^n(y))^{m-1}(x))\,(\lambda y.\, z^n(y)) =$$
$$= \lambda z.\ (\lambda w.\, (\lambda y.\, w^n(y))^{m-1})\,(\lambda x.\, z^n(x)) =$$
$$= \lambda z.\ (\lambda y.\, ((\lambda x.\, z^n(x))^n(y))^{m-1}) =$$
$$= \lambda z.\ (\lambda y.\, ((z^n(y))^n)^{m-1}) =$$
$$= \lambda zy.\, ((z^n(y))^n)^{m-1}$$

$$A_{exp}\, c_n\, c_m = c_m\, c_n = (\lambda fx.\, f^m(x))\,(\lambda ab.\, a^n(b)) =$$
$$= \lambda x.\ (\lambda ab.\, a^n(b))^m(x) =$$
$$= \lambda x.\ (\lambda ab.\, a^n(b))\,((\lambda ab.\, a^n(b))^{m-1}(x)) =$$
$$= \lambda w.\ (\lambda ab.\, a^n(b))\,((\lambda t.\, w^n(t))^{m-1}) =$$
$$= \lambda w.\ (\lambda b.\, ((\lambda t.\, w^n(t))^{m-1})^n(b)) =$$
$$= \lambda w.\ (\lambda b.\, ((w^n(b))^{m-1})^n) =$$
$$= \lambda zy.\, ((z^n(y))^n)^{m-1}$$

$$\Rightarrow\ A_{exp}\, c_n\, \underline{\underline{c_m}} = \underline{\underline{A_*}}\,(A_{exp}\, c_n\, c_{m-1})\, c_n \qquad !!!$$

$$A_{exp}\, c_n\, c_1 = c_1\, c_n = (\lambda fx.\, f(x))\,(\lambda zy.\, z^n(y)) =$$
$$= \lambda x.\ (\lambda zy.\, z^n(y))(x) =$$
$$= \lambda x.\ (\lambda y.\, x^n(y)) =$$
$$= \lambda ft.\, f^n(t) = \underline{\underline{c_n}}$$

$I \equiv \lambda x.x, \quad K \equiv \lambda xy.x, \quad K_* \equiv \lambda xy.y, \quad S \equiv \lambda xyz.xz(yz)$

$\text{true} \equiv K, \quad \text{false} \equiv K_*$

$[M,N] = \lambda z.zMN$

$[M,N]\,\text{true} = (\lambda z.zMN)\,\text{true} = KMN = M$

$[M,N]\,\text{false} = N$

$\ulcorner 0 \urcorner \equiv I, \quad \ulcorner n+1 \urcorner \equiv [\text{false}, \ulcorner n \urcorner] = \lambda z.z\,K_*\,\ulcorner n \urcorner$

$S^+ \equiv \lambda x.[\text{false}, x] = \lambda x.(\lambda z.z\,K_*\,x)$

$P^- \equiv \lambda x.x\,\text{false}$

$\text{zero} \equiv \lambda x.x\,\text{true}$

$P^-\,\ulcorner n+1 \urcorner = (\lambda x.x\,\text{false})(\lambda z.z\,K_*\,\ulcorner n \urcorner) =$

$= (\lambda z.z\,K_*\,\ulcorner n \urcorner)\,K_* = K_*\,K_*\,\ulcorner n \urcorner = \underline{\underline{\ulcorner n \urcorner}}$

$\text{zero}\,\ulcorner 0 \urcorner = (\lambda x.x\,\text{true})(\lambda x.x) =$

$= (\lambda x.x)\,\text{true} = \text{true}$

$\text{zero}\,\ulcorner n+1 \urcorner = (\lambda x.x\,\text{true})(\lambda z.z\,K_*\,\ulcorner n \urcorner) =$

$= (\lambda z.z\,K_*\,\ulcorner n \urcorner)\,K = K\,K_*\,\ulcorner n \urcorner = \underline{\underline{\text{false}}}$

16.10.2008 TH
PETER ČERNO

NAIL078 LAMBDA-KALKULUS I

PRIMITÍVNE / ČIASTOČNE REKURZÍVNE FUNKCIE

ZÁKLADNÉ FUNKCIE:

(1) $o(x) \simeq 0$

(2) $s(x) \simeq x+1$

(3) $p_n^j(x_1, \dots, x_n) \simeq x_j$

SÚ TOTÁLNE (VŠADE DEFINOVANÉ)

ODVODZOVACIE PRAVIDLÁ:

(1) OPERÁTOR SUBSTITÚCIE:

$$S_n(f, g_1, \dots, g_m) \equiv h$$

$$h(x_1, \dots, x_n) \simeq f(g_1(x_1, \dots, x_n), \dots, g_m(x_1, \dots, x_n))$$

(2) OPERÁTOR PRIMITÍVNEJ REKURZIE:

$$\underbrace{R_n(f, g)}_{n-1 \text{ PREM.}} \equiv \underbrace{h}_{n \text{ PREM.}}$$

$$h(0, x_2, \dots, x_n) \simeq f(x_2, \dots, x_n)$$

$$h(y+1, x_2, \dots, x_n) \simeq g(y, h(y, x_2, \dots, x_n), x_2, \dots, x_n)$$

(3) OPERÁTOR MINIMALIZÁCIE

$$\underbrace{M_n(f)}_{n+1 \text{ PREM.}} \equiv \underbrace{h}_{n \text{ PREM.}}$$

$$h(x_1, \dots, x_n) = \mu_y(f(x_1, \dots, x_n, y) \simeq 0)$$

PRIMITÍVNE REKURZÍVNE FCIE:

ZÁKL. AXIÓMY + ODV. PRAVIDLÁ (1)+(2)

ČIASTOČNE REKURZÍVNE FCIE:

ZÁKL. AXIÓMY + ODV. PRAVIDLÁ (1)+(2)+(3)

$f : \mathbb{N}^p \to \mathbb{N}$ IS λ-DEFINABLE $\Leftrightarrow$ $\exists F \in \Lambda$

$$F \ulcorner n_1 \urcorner \ldots \ulcorner n_p \urcorner = \ulcorner f(n_1, \ldots, n_p) \urcorner$$

f IS λ-DEFINED BY F.

<u>THEOREM</u>: ALL RECURSIVE FUNCTIONS ARE λ-DEFINABLE.

<u>BASIC FUNCTIONS</u>:

$Z \equiv \lambda x. \ulcorner 0 \urcorner \qquad S^+ \equiv \lambda x. [\text{false}, x]$,

$P_n^i \equiv \lambda x_1 \ldots x_n \cdot x_i$

<u>OPERATIONS</u>:

(1) COMPOSITION $g, h_1, \ldots, h_m$ BE λ-DEFINED BY $G, H_1, \ldots, H_m$.

THEN $f(\vec{n}) = g(h_1(\vec{n}), \ldots, h_m(\vec{n}))$

IS λ-DEF. BY $F \equiv \lambda \vec{x}. G(H_1 \vec{x}) \ldots (H_m \vec{x})$

(2) PRIMITIVE RECURSION

LET f BE DEFINED BY

$$f(0, \vec{n}) = g(\vec{n}), \quad f(k+1, \vec{n}) = h(f(k, \vec{n}), k, \vec{n})$$

g, h BE λ-DEF. BY G, H RESP.

$$F x \vec{Y} \equiv \text{if } (\text{zero } x) \text{ then } G \vec{Y} \text{ else } H(F(P^- x) \vec{Y})(P^- x) \vec{Y}$$

CONTEXT $D[f, x, \vec{Y}] =$

$$(\text{zero } x)(G \vec{Y})(H(f(P^- x) \vec{Y})(P^- x) \vec{Y})$$

$\rightsquigarrow$ CONTEXT-LEMMA : $\exists F$, $\underline{\underline{F = Y(\lambda f x \vec{Y}. D[f, x, \vec{Y}])}}$

16.10.2008 Th
PETER ČERNO

NAIL078 LAMBDA-KALKULUS I

$$Y = \lambda f.\ (\lambda x.\ f(xx))(\lambda x.\ f(xx))$$

CONTEXT $D[f, x, \vec{y}] = (\text{zero } x)(G\vec{y})(\ \dots\ )$

$$T := \lambda f x \vec{y}.\ D[f, x, \vec{y}]$$

$$F := YT = \boxed{(\lambda w.\ (\lambda f x \vec{y}.\ D[f, x, \vec{y}])(ww))(\lambda w.(\ \dots\ ))}$$

$$T(YT) = YT \Rightarrow F = TF = \lambda x \vec{y}.\ D[F, x, \vec{y}]$$

$$\Rightarrow F x \vec{y} = D[F, x, \vec{y}]$$

(3) (REGULAR) MINIMALIZATION

$$f(\vec{n}) = \mu_m\,(g(\vec{n}, m) = 0) \qquad \underline{\forall \vec{n}\ \exists m\ g(\vec{n}, m) = 0}$$

CONTEXT-LEMMA $\Rightarrow \exists H$:

$$H = \text{if } (\text{zero } (G\vec{x}y)) \text{ then } y \text{ else } H\vec{x}(S^+ y)$$

$$F := \underline{\underline{\lambda \vec{x}}}.\ H\vec{x}[0]$$

DOUBLE FIXED POINT THEOREM

$$\forall A\ \forall B\ \exists X\ \exists Y : X = AXY,\ Y = BXY$$

$$\boxed{F := \lambda x.\ [A(x\ \text{true})(x\ \text{false}),\ B(x\ \text{true})(x\ \text{false})]}$$

$$\Rightarrow \exists Z : FZ = Z$$

$$\boxed{X := Z\ \text{true},\quad Y := Z\ \text{false}}$$

$$X = Z\ \text{true} = (FZ)\ \text{true} = A(Z\ \text{true})(Z\ \text{false}) = AXY$$

FOR A, B WE HAVE

$$X = Y(\lambda x.[\overbrace{A(x\ \text{true})(x\ \text{false}),\ B(x\ \text{true})(x\ \text{false})}^{F}])\ \text{true}$$
$$Y = Y(\quad -\|- \quad)\ \text{false}$$

$$X = L\,AB\ ,\quad Y = R\,AB$$

$$L = \lambda ab.(Y(\lambda x.[a(x\ \text{true})(x\ \text{false}),\ b(x\ \text{true})(x\ \text{false})]))\ \text{true}$$
$$R = \lambda ab.(\quad -\|- \quad)\ \text{false}$$

$$L\,AB = A\,L\,AB\,R\,AB$$
$$R\,AB = B\,L\,AB\,R\,AB$$

GIVEN CONTEXTS $C_1[f, g\ x]$, $C_2[f, g, x]$

$$F_1 := L\,(\lambda fgx.\ C_1[f,g,x])\,(\lambda fgx.\ C_2[f,g,x])$$
$$F_2 := R\,\underbrace{(\lambda fgx.\ C_1[f,g,x])}_{A}\,\underbrace{(\lambda fgx.\ C_2[f,g,x])}_{B}$$

$$F_1 = L\,AB = A\,L\,AB\,R\,AB = A\,F_1\,F_2 =$$
$$= \lambda x.\ C_1[F_1, F_2, x]$$

$$F_2 = R\,AB = B\,L\,AB\,R\,AB = B\,F_1\,F_2 =$$
$$= \lambda x.\ C_2[F_1, F_2, x]$$

24.10.2008 Fr
PETER ČERNO

NAIL078 LAMBDA KALKULUS I

KLEENE: THERE IS AN „INTERPRETER" COMBINATOR E: $\forall$ λ-CLOSED TERM M WITHOUT CONSTANTS: $E\ulcorner M\urcorner = M$

$\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ RECURSIVE BIJECTION

$\# v^{(n)} = \langle 0, n \rangle$, $\# c^{(n)} = \langle 1, n \rangle$,

$\#(MN) = \langle 2, \langle \# M, \# N \rangle \rangle$, $\#(\lambda x.M) = \langle 3, \langle \# x, \# M \rangle \rangle$

$\ulcorner M \urcorner := \ulcorner \# M \urcorner$

$$\langle x, y \rangle = \boxed{\frac{(x+y)(x+y+1)}{2} + x}$$

y \ x	0	1	2	3	4	5	6
5	15						
4	10	16					
3	6	11	17				
2	3	7	12	18			
1	1	4	8	13	19		
0	0	2	5	9	14	20	

$$\frac{x^2 + xy + x + yx + y^2 + y}{2} + x = C$$

$$\frac{A.(A+1)}{2} \leq C \leq \frac{A(A+1)}{2} + A$$

$$\boxed{x := C - \frac{A(A+1)}{2}, \quad y := A - x}$$

$$A(A+1) \leq 2C \leq A(A+3)$$

$$\left(A + \tfrac{1}{2}\right)^2 - \tfrac{1}{4} \leq 2C \leq \left(A + \tfrac{3}{2}\right)^2 - \tfrac{9}{4}$$

$$\boxed{A := \left\lfloor \sqrt{2C + \tfrac{1}{4}} - \tfrac{1}{2} \right\rfloor}$$

MÔŽEME PREDPOKLADAŤ, ŽE $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$ JE λ-DEFINOVATEĽNÁ, TJ. EXISTUJE $\diamond \in \Lambda$ TAKÉ, ŽE $\diamond \ulcorner n_1 \urcorner \ulcorner n_2 \urcorner = \ulcorner \langle n_1, n_2 \rangle \urcorner$ PRE $\forall n_1, n_2 \in \mathbb{N}$

ĎALEJ NECH $l : \mathbb{N} \to \mathbb{N}$, $r : \mathbb{N} \to \mathbb{N}$ SÚ REKURZÍVNE $l^\diamond \langle n_1 n_2 \rangle = n_1$, $r^\diamond \langle n_1, n_2 \rangle = n_2$, $L^\diamond, R^\diamond$ SÚ PRÍSL. λ-TERMY.

NAKONIEC NECH $CMP \in \Lambda$

$$CMP\ \ulcorner n_1 \urcorner\ \ulcorner n_2 \urcorner = \begin{cases} true & \Leftrightarrow\ n_1 = n_2 \\ false & \Leftrightarrow\ n_1 \neq n_2 \end{cases}$$

HĽADÁME:

$$CMP\ x\ y = \text{if zero } x \text{ then zero } y \text{ else } (\text{if zero } y \text{ then zero } x \text{ else } CMP\ (P^- x)(P^- y))$$

DEFINUJME KONTEXT:

$$D[f, x, y] := (\text{zero } x)((\text{zero } y)((\text{zero } y)(\text{zero } x)(f\ (P^- x)(P^- y))))$$

MÁME KOMBINÁTOR $Y = \lambda f.(\lambda x. f(xx))(\lambda x. f(xx))$

POLOŽME $\underline{\underline{T := \lambda f x y.\ D[f, x, y]}}$

POTOM $\underline{\underline{CMP = YT}}$

PRETOŽE PLATÍ $CMP = T\,CMP = \lambda x y.\ D[CMP, x, y]$

(1) CHCEME SKONŠTRUOVAŤ E_1 TAKÉ, ŽE PRE $\forall F$

$$E_1\ F\ulcorner x \urcorner = F\ulcorner x \urcorner$$

$$E_1\ F\ulcorner MN \urcorner = (E_1\ F\ulcorner M \urcorner)(E_1 F\ulcorner N \urcorner)$$

$$E_1\ F\ulcorner \lambda x.M \urcorner = \lambda z.(E_1\ F_{[x:=z]}\ulcorner M \urcorner)$$

KDE $F_{[x:=z]}\ulcorner n \urcorner = \begin{cases} F\ulcorner n \urcorner & \text{AK } \# n \neq \# x \\ z & \text{AK } \# n = \# x \end{cases}$

24.10.2008 Fr
PETER ČERNO

NAIL078 LAMBDA KALKULUS I

$F_{[x:=z]}$ ZAPÍŠEME AKO $G\lceil x \rceil z F$

KDE G DEFINUJEME AKO:

$$
\begin{aligned}
G = \lambda x z f n . (& \\
&\text{if } \mathrm{CMP}\ n\ x \text{ then } z \\
&\text{else } (f\ n)\ )
\end{aligned}
$$

E_1 POTOM MOŽNO SKONŠTRUOVAŤ AKO

$$
\begin{aligned}
E_1 = \lambda f x . (& \\
&\text{if } \mathrm{CMP}\ 0\ (L^\diamond x) \text{ then } (f\ x) \\
&\text{else if } \mathrm{CMP}\ 1\ (L^\diamond x) \text{ then } (f\ x) \\
&\text{else if } \mathrm{CMP}\ 2\ (L^\diamond x) \text{ then} \\
&\quad (E_1\ f\ (L^\diamond (R^\diamond x)))(E_1\ f\ (R^\diamond (R^\diamond x))) \\
&\text{else if } \mathrm{CMP}\ 3\ (L^\diamond x) \text{ then} \\
&\quad \lambda z . (E_1\ \boxed{G\ (L^\diamond (R^\diamond x))\ z\ f}\ \boxed{R^\diamond (R^\diamond x)}) \\
)&
\end{aligned}
$$

Z KONTEXTOVÉHO LEMMY VYPLÝVA, ŽE TAKÉ E_1 EXISTUJE.

DOKÁŽEME INDUKCIOU PODĽA ZLOŽITOSTI M, ŽE

$$E_1 F \lceil M \rceil = M[x_1 := F\lceil x_1 \rceil, \ldots, x_n := F\lceil x_n \rceil][c_1 := F\lceil c_1 \rceil, \ldots, c_m := F\lceil c_m \rceil]$$

(SUBSTITÚCIA)

KDE $\{x_1, \ldots, x_n\} = FV(M)$,
$\{c_1, \ldots, c_m\}$ SÚ $\forall$ KONŠTANTY V M.

DŮKAZ:

KEĎ JE M PREMENNÁ ALEBO KONŠTANTA:

$$E_1 F\lceil M\rceil = E_1 F\lceil x\rceil = F\lceil x\rceil = x[x := F[x]] \quad \checkmark$$

ĎALEJ

$$E_1 F\lceil MN\rceil = (E_1 F\lceil M\rceil)(E_1 F\lceil N\rceil) =$$

$$= (M[x_1 := F\lceil x_1\rceil, \dots, x_n := F\lceil x_n\rceil][c_1 := F\lceil c_1\rceil, \dots, c_m := F\lceil c_m\rceil])$$

$$(N[\quad -\|- \quad][\quad -\|- \quad])$$

$$= (MN)[\quad -\|- \quad][\quad -\|- \quad]$$

NAKONIEC $\checkmark$

$$E_1 F\lceil \lambda x.M\rceil = \boxed{\lambda z.(E_1(F_{[x:=z]})\lceil M\rceil)}$$

KDE $F_{[x:=z]} = G\lceil x\rceil z F =$

$\lambda n.(\text{if } \mathrm{cmp}\ n\ \lceil x\rceil \text{ then } z \text{ else } F\ n)$

$$= \lambda z.(M[x_1 := F_{[x:=z]}\lceil x_1\rceil, \dots, x_n := F_{[x:=z]}\lceil x_n\rceil] \quad (*)$$
$$[c_1 := F_{[x:=z]}\lceil c_1\rceil, \dots, c_m := F_{[x:=z]}\lceil c_m\rceil])$$

AK x NEMÁ VOĽNÝ VÝSKYT V M, TJ. $x \notin \{x_1, \dots, x_n\}$, POTOM

$$(\lambda x.M)[x_1 := F\lceil x_1\rceil, \dots x_n := F\lceil x_n\rceil][c_1 \dots c_m] =$$

$$= \lambda z.(M[x_1 := F\lceil x_1\rceil, \dots, x_n := F\lceil x_n\rceil][c_1 \dots c_m]) = (*)$$

$$F_{[x:=z]}\lceil x_i\rceil = F\lceil x_i\rceil \quad \# x_i \neq \# x$$

$$F_{[x:=z]}\lceil c_j\rceil = F\lceil c_j\rceil \quad \# c_j \neq \# x$$

24.10.2008 Fr
PETER ČERNO

NAIL078 LAMBDA KALKULUS I

NECH x MÁ VOĽNÝ VÝSKYT V M, BÚNO $x = x_1$.

POTOM

$$\lambda z.\big(M[x_1 := F_{[x_1:=z]}\lceil x_1\rceil, \dots, x_n := F_{[x_1:=z]}\lceil x_n\rceil]$$
$$[c_1 := F_{[x_1:=z]}\lceil c_1\rceil, \dots, c_m := F_{[x_1:=z]}\lceil c_m\rceil]\big)$$

$$= \lambda z.\big(M[x_1 := z,\ x_2 := F\lceil x_2\rceil, \dots, x_n := F\lceil x_n\rceil]$$
$$[c_1 := F\lceil c_1\rceil, \dots, c_m := F\lceil c_m\rceil\big) =$$

$$= (\lambda x_1.M)[x_2 := F\lceil x_2\rceil, \dots, x_n := F\lceil x_n\rceil]$$
$$[c_1 := F\lceil c_1\rceil, \dots, c_m := F\lceil c_m\rceil]$$

✓

NA DOKONČENIE DOKAZU STAČÍ VZIAŤ

$$\underline{\underline{E = \lambda x.\, E_1\, I\, x}}$$

$\square$

$\langle\cdot,\cdot\rangle : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$, $L^\Delta, R^\Delta : \mathbb{N} \to \mathbb{N}$ SÚ REKURZÍVNE,

$\Rightarrow$ $\# : \Lambda \to \mathbb{N}$ JE TAKMEŽ REKURZÍVNA

$\Rightarrow$ $AP(\#M, \#N) = \#(MN) = \langle 2, \langle \#M, \#N\rangle\rangle$

$Num(n) = \#\lceil n\rceil$

SÚ REKURZÍVNE $\Rightarrow$ λ-DEFINOVATEĽNÉ

$AP\lceil M\rceil\lceil N\rceil = \lceil MN\rceil$

$Num\lceil n\rceil = \lceil\lceil n\rceil\rceil$

$$\underline{\underline{AP = \lambda mn.\big(\diamond\lceil 2\rceil(\diamond\, m\, n)\big)}}$$

Num MOŽNO DEFINOVAŤ AKO

$\lambda n.($

if Zero n then $\ulcorner \lambda x.x \urcorner = \longrightarrow \Diamond \ulcorner 3 \urcorner (\Diamond \ulcorner \# x \urcorner \ulcorner \# x \urcorner)$

$= \ulcorner \langle 3, \langle \# x, \# x \rangle \rangle \urcorner$ } KONKRÉTNY λ-TERM PRE DANÉ $\langle \cdot, \cdot \rangle$ A $\#$

else

$// \; n = S^+(P^- n) = [\text{false}, P^- n] =$

$= \lambda z.z \; \text{false} \; (P^- n)$

$\Rightarrow \ulcorner n \urcorner = \ulcorner \lambda z.((z \; \text{false})(P^- n)) \urcorner =$

$= \ulcorner \langle 3, \langle \# z, ((z \; \text{false})(P^- n)) \rangle \rangle \urcorner =$

$= \ulcorner \langle 3, \langle \# z, \langle 2, \langle \underbrace{z \; \text{false}}_{\langle 2, \langle \# z, \text{false} \rangle \rangle}, (P^- n) \rangle \rangle \rangle \rangle \urcorner$

$\rightarrow \Diamond \ulcorner 3 \urcorner (\Diamond \ulcorner \# z \urcorner (\Diamond \ulcorner 2 \urcorner (\Diamond \ulcorner z \; \text{false} \urcorner (\text{Num}(P^- n)))))$

Z CONTEXT-LEMMY UŽ VYPLÝVA EXISTENCIA Num

AK BY SME MALI λ-FUNKCIU $F : \Lambda \to \Lambda$

$M \mapsto \ulcorner M \urcorner$

POTOM BY Num BOLO $\lambda n.(F \, n)$

$\boxed{?}$

OTÁZKA ZNIE, ČI TAKÁ F VÔBEC EXISTUJE :)

ZO SECOND FIXED POINT THEOREM BY SME NAŠLI X TAKÉ, ŽE $F \ulcorner X \urcorner = X$, TJ. $\ulcorner \ulcorner X \urcorner \urcorner = X$

HMM :)

24.10.2008 Fr
PETER ČERNO

NAIL078 LAMBDA KALKULUS I

SECOND FIXED POINT THEOREM

$$\boxed{\forall F \exists X \; F\ulcorner X \urcorner = X}$$

$$\boxed{\begin{aligned} W &\equiv \lambda x.F(AP\ x\ (Num\ x)) \\ X &\equiv W\ulcorner W \urcorner \end{aligned}}$$

$$\begin{aligned} X &= (\lambda x.F(AP\ x\ (Num\ x)))\ulcorner W \urcorner = \\ &= F(AP\ulcorner W \urcorner\ (Num\ulcorner W \urcorner)) = \\ &= F\ulcorner W\ulcorner W \urcorner \urcorner = F\ulcorner X \urcorner \end{aligned}$$

$\square$

USEFUL IN PROVING UNDECIDABILITY RESULTS.

THEOREM (SCOTT)

$A \subseteq \Lambda$

(1) $A \neq \emptyset \wedge A \neq \Lambda$

(2) $M \in A,\ M = N \Rightarrow N \in A$

$\Rightarrow \# A = \{\# M \mid M \in A\}$ IS NOT RECURSIVE

SPOROM:

A JE REKURZIVNA $\Rightarrow$ EXISTUJE UZAVRETÝ λ-TERM F TAKÝ, ŽE

$M \in A \Leftrightarrow F\ulcorner M \urcorner = \ulcorner 0 \urcorner \qquad M \notin A \Leftrightarrow F\ulcorner M \urcorner = \ulcorner 1 \urcorner$

NECH $M_0 \in A$, $M_1 \notin A$. POLOŽME

$$\boxed{G = \lambda x.(\text{if } Zero(Fx) \text{ then } M_1 \text{ else } M_0)}$$

SECOND FIXED POINT THEOREM: $\exists M: G\ulcorner M \urcorner = M$

$M \in A \Leftrightarrow F\ulcorner M \urcorner = \ulcorner 0 \urcorner \Rightarrow G\ulcorner M \urcorner = M_1 = M \notin A$

$M \notin A \Leftrightarrow F\ulcorner M \urcorner = \ulcorner 1 \urcorner \Rightarrow G\ulcorner M \urcorner = M_0 = M \in A$ $\lightning$

M IS IN NORMAL FORM $\Leftrightarrow$ M HAS NO PART (REDEX) OF THE FORM $(\lambda x.P)Q$

THEOREM (CHURCH, SCOTT)

$NF = \{M \mid M$ HAS A NORMAL FORM$\}$ IS NOT RECURSIVE

NF IS CLOSED UNDER $=$, $NF \neq \emptyset$,

$\Omega = (\lambda x.xx)(\lambda x.xx) \notin NF \Rightarrow NF \neq \Lambda$ $\square$

26. 10. 2008 Su
PETER ČERNO

NAIL078 LAMBDA KALKULUS I

SÉMANTIKA λ-KALKULU

DÁVA VÝZNAM VÝRAZOM V L { OPERATIONAL SEMANTICS / DENOTATIONAL SEMANTICS

OPERATIONAL SEMANTICS

$R \subseteq \Lambda \times \Lambda$

(i) COMPATIBLE: $M R N \Rightarrow ZM R ZN, \ldots$ $\quad \rightarrow_\beta$

(ii) REDUCTION ON Λ: COMPATIBLE + REFL. & TRANS. $\quad \twoheadrightarrow_\beta$

(iii) CONGRUENCE ON Λ: COMPATIBLE + EQUIV. $\quad =_\beta$

(i) (1) $(\lambda x.M)N \rightarrow_\beta M[x:=N]$ (β-REDUCTION)
(2) COMPATIBILITY PROPERTIES

(ii) REFLEXIVE + TRANSITIVE CLOSURE OF $\rightarrow_\beta$

(iii) EQUIVALENCE RELATION GENERATED BY $\rightarrow_\beta$

$M \rightarrow_\beta N$: M β-REDUCES TO N IN ONE STEP

$\twoheadrightarrow_\beta$: M β-REDUCES TO N

$=_\beta$: M IS β-CONVERTIBLE TO N

THEOREM: $M =_\beta N \Leftrightarrow \lambda \vdash M = N$

$\Rightarrow$) RELÁCIU $=_\beta$ MÔŽEME DEFINOVAŤ AKO

(1) $M \rightarrow_\beta N \Rightarrow M =_\beta N$
(2) $M =_\beta M$
(3) $M =_\beta N \Rightarrow N =_\beta M$
(4) $M =_\beta N \wedge N =_\beta L \Rightarrow M =_\beta L$

ĎALEJ IDEME INDUKCIOU PODĽA POČTU n KROKOV (1)-(4), KTORÉ POTREBUJEME K ODVODENIU $M =_\beta N$.

$\Leftarrow$) INDUKCIA PODĽA DĹŽKY DÔKAZU.

LEMMA: M IS β-NORMAL, $M \rightarrow_\beta N \Rightarrow M \equiv N$

CHURCH-ROSSER THM.

$$\begin{tikzcd} & M \arrow[dl, "\beta"'] \arrow[dr, "\beta"] & \\ M_1 & & M_2 \end{tikzcd} \Rightarrow \exists N: \begin{tikzcd} M_1 \arrow[dr, "\beta"'] & & M_2 \arrow[dl, "\beta"] \\ & N & \end{tikzcd}$$

<u>COROLLARY</u> : $M_1 =_\beta M_2 \Rightarrow \exists N : M_1 \twoheadrightarrow_\beta N,\ M_2 \twoheadrightarrow_\beta N$

INDUCTION ON GENERATION OF $=_\beta$

(4) $M_1 =_\beta L \wedge L =_\beta M_2$ BY INDUCTION:

$$\begin{tikzcd} M_1 \arrow[dr] & L \arrow[dl] \arrow[dr] & M_2 \arrow[dl] \\ N_1 & & N_2 \end{tikzcd} \xrightarrow{\text{CHURCH-ROSSER}} \begin{tikzcd} N_1 \arrow[dr] & & N_2 \arrow[dl] \\ & N & \end{tikzcd}$$

. $\square$

<u>COROLLARY</u> : (i) IF N IS A β-NORMAL FORM OF $M \Rightarrow M \twoheadrightarrow_\beta N$.

$M =_\beta N \Rightarrow \exists L : M \twoheadrightarrow_\beta L,\ N \twoheadrightarrow_\beta L \Rightarrow N \equiv L$.

(ii) AT MOST 1 β-NORMAL FORM :

$N_1 =_\beta N_2 =_\beta M \Rightarrow N_1 \twoheadrightarrow_\beta L,\ N_2 \twoheadrightarrow_\beta L \Rightarrow N_1 \equiv N_2$.

$\lambda \nvdash$ true = false (DISTINCT β-NORMAL FORMS)

<u>TURING FIXED POINT COMBINATOR</u>

$$\boxed{\Theta = (AA),\quad A = \lambda xy.\, y(xxy)}$$

$$\Theta F \equiv (AA)F \to_\beta (\lambda y.\, y((AA)y))F \to_\beta F((AA)F) \equiv F(\Theta F)$$

$\square$

DOUBLE FIXED POINT THEOREM

$L = \lambda ab.\, (\Theta(\lambda x.\, [a(x\ \text{true})(x\ \text{false}),\ b(x\ \text{true})(x\ \text{false})]))\ \text{true}$

$R = \lambda ab.\, (\quad \text{—''—} \quad)\ \text{false}$

$L \equiv \lambda ab.\, (\Theta(\overbrace{\lambda x.[a(x\,\text{true})(x\,\text{false}),\ b(x\,\text{true})(x\,\text{false})]}^{F}))\ \text{true} \twoheadrightarrow_\beta$

$\lambda ab.\, (F(\Theta F))\ \text{true} \twoheadrightarrow_\beta \lambda ab.$

$([a((\Theta F)\,\text{true})((\Theta F)\,\text{false}),\ b((\Theta F)\,\text{true})((\Theta F)\,\text{false}))])\ \text{true}$

$\twoheadrightarrow_\beta \lambda ab.\, (a((\Theta F)\,\text{true})((\Theta F)\,\text{false}))$

$\boxed{?}$

06. 01. 2009 Tu
PETER ČERNO

NAIL078 LAMBDA-CALCULUS I

2.1. (i)

$$M_1 \equiv Y(\lambda x.xY(\lambda zw.Yz))$$
$$Y(\lambda x.(xY)(\lambda z.(\lambda w.Yz)))$$
$$(Y(\lambda x.((xY)(\lambda z.(\lambda w.(Yz))))))$$
$$(v'(\lambda v.((vv')(\lambda v''.(\lambda v'''.(v'v''))))))$$

(ii) $\lambda v'.(\lambda v''.((((\lambda v.v)v')v'')((v''(\lambda v'''.(v'v''')))v'')))$

$$\lambda Y.\ \lambda Z.\ (\lambda x.x)YZ\ (Z(\lambda W.YW)\ Z)$$

$$\lambda YZ.(\lambda x.x)YZ(Z(\lambda W.YW)Z)$$

2.2. SUBSTITUTION LEMMA

$x \not\equiv Y,\ x \notin FV(L)$ (PREDP., ŽE X, Y NIE SÚ NIKDE VIAZ.)

$$M[x:=N][Y:=L] \equiv M[Y:=L][x:=N[Y:=L]]$$

INDUKCIA PODĽA ZLOŽITOSTI M

a) $x[x:=N][Y:=L] \equiv N[Y:=L]$

$x[Y:=L][x:=N[Y:=L]] \equiv x[x:=N[Y:=L]] \equiv N[Y:=L]$

b) $Y[x:=N][Y:=L] \equiv Y[Y:=L] = L$

$Y[Y:=L][x:=N[Y:=L]] = L[x:=N[Y:=L]] = L$

c) $z[\ldots][\ldots] \equiv z \equiv z[\ldots][\ldots]$

d) $(M_1 M_2)[x:=N][y:=L] \equiv$

$\equiv ((M_1[x:=N])(M_2[x:=N]))[y:=L] \equiv$

$\equiv ((M_1[x:=N][y:=L])(M_2[x:=N][y:=L])) \equiv$ IH

$\equiv ((M_1[y:=L][x:=N[y:=L]])(\ \dots\ )) \equiv$

$\equiv \dots \equiv (M_1 M_2)[y:=L][x:=N[y:=L]]$

e) $(\lambda z.M)[x:=N][y:=L] \equiv$

$\equiv \lambda z.M[x:=N][y:=L] \equiv$ IH

$\equiv \lambda z.M[y:=L][x:=N[y:=L]] \equiv$

$\equiv (\lambda z.M)[y:=L][x:=N[y:=L]]$

2.3. (i)

$\lambda \vdash M_1 = M_2 \Rightarrow \lambda \vdash \lambda x.M_1 = \lambda x.M_2 \Rightarrow$

$\lambda \vdash (\lambda x.M_1)N = (\lambda x.M_2)N$

$\lambda \vdash M_1[x:=N] = M_2[x:=N]$

(ii) $\lambda \vdash M_1 = M_2$ & $\lambda \vdash N_1 = N_2$

$\Rightarrow \lambda \vdash M_1[x:=N_1] = M_2[x:=N_1]$ (i)

$\lambda \vdash N_1 = N_2 \Rightarrow \lambda \vdash (\lambda x.M_2)N_1 = (\lambda x.M_2)N_2$,

$\lambda \vdash M_2[x:=N_1] = M_2[x:=N_2]$

$\lambda \vdash M_1[x:=N_1] = M_2[x:=N_2]$

06.01.2009 Tu
PETER ČERNO

NAIL078 LAMBDA-CALCULUS I

2.4. $c_n = \lambda f x.\, f^n(x)$

(i) $A_+ = \lambda xypq.\, xp(ypq)$

$$A_+ c_n c_m = \lambda pq.\, c_n p (c_m pq) =$$
$$= \lambda pq.\, c_n p (p^m(q)) = \lambda pq.\, p^n(p^m(q)) = \underline{\underline{c_{n+m}}}.$$

(ii) $A_* = \lambda xyz.\, x(yz)$

$$A_* c_n c_m = \lambda z.\, c_n (c_m z) = \lambda z.\, c_n(\lambda x.\, z^m(x)) =$$
$$= \lambda z.\, \lambda y.\, \underbrace{(\lambda x.\, z^m(x))^n (y)}_{(c_m z)^n (y) = z^{m*n}(y)} = \underline{\underline{c_{m*n}}}.$$

2.6. (iii)

$$SKSKSK = KK(SK)SK = KSK = S$$

2.7. (i) $KI = (\lambda xy.\, x)(\lambda z.\, z) = \lambda y.\, (\lambda z.\, z) = K_*$

(ii) $SKK = (\lambda xyz.\, xz(yz))KK =$
$= \lambda z.\, Kz(Kz) = \lambda z.\, z = I$

2.8. (i) $\lambda xy.\, x(yx)y$

(ii) $M, N, L \in \Lambda^0$ $\quad FMNL = N(\lambda x.\, M)(\lambda yz.\, y\, LM)$

$F \equiv \lambda uvw.\, v(\lambda x.\, u)(\lambda yz.\, ywu)$

2.9. (i) $Fx = xI \qquad F = \lambda x.\, (xI)$

(ii) $Fxy = xIy \qquad F = \lambda xy.\, (xIy)$

2.10. (i) $Fx = F \Leftarrow F = \lambda x.\, F = (\lambda fx.\, f)F \Leftarrow$

$\Leftarrow F = Y(\lambda fx.\, f)$

$Y \equiv \lambda f.\, (\lambda x.\, f(xx))(\lambda x.\, f(xx))$

$$\underline{\underline{F = (\lambda y.\, (\lambda fx.\, f)(yy))(\lambda y.\, (\lambda fx.\, f)(yy))}}$$

$$\boxed{\begin{aligned} Fx &= (Y(\lambda fx.\, f))x \\ &= ((\lambda fx.\, f)(Y(\cdots)))x \\ &= (\lambda x.\, F)x = \\ &= \underline{\underline{F}} \end{aligned}}$$

(ii) $Fx = xF \Leftarrow F = \lambda x.(xF) = (\lambda fx.(xf))F$

$\Leftarrow F = Y(\lambda fx.xf)$

$Fx = (Y(\lambda fx.xf))x = ((\lambda fx.xf)F)x =$

$= (\lambda x.xF)x = xF$

(iii) $FIKK = FK \Leftarrow F = \lambda xyz.Fy = (\lambda fxyz.fy)F$

$F = Y(\lambda fxyz.fy)$

2.11. $\forall C[,]\ \exists F\ \forall \vec{x}\ F\vec{x} = C[F, \vec{x}]$

$F\vec{x} = C[F, \vec{x}] \Leftarrow F = \lambda\vec{x}.C[F, \vec{x}] = (\lambda f\vec{x}.C[f, \vec{x}])F$

$\Leftarrow \underline{\underline{F = Y(\lambda f\vec{x}.C[f, \vec{x}])}}$

$F\vec{x} = (Y(\lambda f\vec{x}.C[f, \vec{x}]))\vec{x} = ((\lambda f\vec{x}.C[f, \vec{x}])F)\vec{x} =$

$= (\lambda\vec{x}.C[F, \vec{x}])\vec{x} = \underline{\underline{C[F, \vec{x}]}}$.

2.12. (i) $\Rightarrow$) ZREJMÉ

$\Leftarrow$) $\lambda + (..) \vdash K = K_* \Rightarrow$

$\lambda + (..) \vdash KMN = K_* MN \Rightarrow$

$\lambda + (..) \vdash M = N$.

(ii) $\lambda + (I = K) \vdash K = I$

$\Rightarrow \lambda + (I = K) \vdash \underbrace{KI}_{K_* = I} = II \Rightarrow K = K_*$.

06.01.2008 T4
PETER ČERNO

NAIL078 LAMBDA-KALKULUS I

2.12. (iii) $FI = x$, $FK = y$

$IK = K$, $KI = K_*$, $SKK = I$, $SKI = I$ ...

$F = \lambda f.\ f I (\lambda uvw.w) K y x$

$FI = II(\lambda uvw.w)Kyx =$
$= I(\lambda uvw.w)Kyx =$
$= x$

$FK = KI(\lambda uvw.w)Kyx =$
$= IKyx = Kyx = y$

(iv) $K \# S$

$K \equiv \lambda xy.x$ $\quad S \equiv \lambda xyz.xz(yz)$

$KK = \lambda y.K = \lambda y.\lambda uv.v$
$KKK = K$

$G = \lambda z.zKK$

$GK = K$, $GS = I$

$F = \lambda f.\ fI(\lambda uvw.w)KK\ K_*$

PLATÍ $FK = K$, $FI = K_*$

$\Rightarrow$ $FGK = K$, $FGS = K_*$

3.1. (i) $\text{Mult}\ \ulcorner m \urcorner\ \ulcorner n \urcorner = \ulcorner m \cdot n \urcorner \qquad \forall m, n \in \mathbb{N}_0$

$$\text{Mult}(0, y) = 0$$
$$\text{Mult}(x+1, y) = \text{Add}(y, \text{Mult}(x, y))$$

$$\text{Mult} = \lambda xy.\ \text{if Zero } x \text{ then } \ulcorner 0 \urcorner$$
$$\text{else Add } y\ (\text{Mult } (P^- x)\ y)$$

$$\Leftarrow \quad \text{Mult} = Y\ \lambda f x y.\ \text{Zero } x\ \ulcorner 0 \urcorner$$
$$\underline{\underline{(\text{Add } y\ (f\ (P^- x)\ y))}}$$

(ii) $\text{Fac}\ \ulcorner n \urcorner = \ulcorner n! \urcorner \qquad \forall n \in \mathbb{N}_0$

$$\text{Fac}(0) = 1$$
$$\text{Fac}(x+1) = \text{Mult}(x+1, \text{Fac}(x))$$

$$\text{Fac} = \lambda x.\ \text{if Zero } x \text{ then } \ulcorner 1 \urcorner$$
$$\text{else Mult } x\ (\text{Fac } (P^- x))$$

$$\Leftarrow \quad \text{Fac} = \underline{\underline{Y}}\ \lambda f x.\ \text{Zero } x\ \ulcorner 1 \urcorner\ (\text{Mult } x\ (f\ (P^- x)))$$

3.2.

$$F = Y\ \lambda f x y.\ \text{if Zero } x \text{ then } S^+ y$$
$$\text{else if Zero } y \text{ then } f\ (P^- x)\ \ulcorner 1 \urcorner$$
$$\text{else } f\ (P^- x)\ (f\ x\ (P^- y))$$

08.01.2009 Th
PETER ČERNO

NAIL078 LAMBDA-KALKULUS I

3.3. $M_0 = x$

$M_{n+1} = M_{n+2} M_n$

$M_1 = M_2 M_0$

$M_2 = M_3 M_1$

$M_3 = M_4 M_2$

$M_4 = M_5 M_3$

$M_5 = M_6 M_4$

...

NAJDIME F TAKÚ, ŽE

$F = \lambda x.\ \text{if Zero } x \text{ then } x$
$\quad \text{else } (F(S^+ x))(F(P^- x))$

A POLOŽME $M_n := F\lceil n \rceil$

3.4. $P_c^- \equiv \lambda xyz.\ x(\lambda pq.\ q(py))(Kz)I \qquad c_n = \lambda fx.\ f^n(x)$

$P_c^- c_{n+1} = \lambda yz.\ (\lambda pq.\ q(py))^{n+1}(Kz)I =$

$= \lambda yz.\ (\lambda pq.\ q(py))^n (\lambda q.\ q((Kz)y))I =$

$= \lambda yz.\ (\lambda pq.\ q(py))^n (\lambda q.\ qz)I$

$(\lambda pq.\ q(py))^{n-1}(\lambda q.\ q((\lambda w.\ wz)y))$
$(\lambda q.\ q(yz))$

$(\lambda pq.\ q(py))^{n-2}(\lambda q.\ q((\lambda w.\ w(yz))y))$
$(\lambda q.\ q(y(yz)))$

...

$= \lambda yz.\ (\lambda q.\ q(y^n(z)))I =$

$= \underline{\underline{\lambda yz.\ y^n(z)}}$

19.05.2009 Tu
PETER ČERNO

NAIL078 LAMBDA-KALKULUS I - SKÚŠKA

FIXED POINT THEOREM :

(i) $\forall F \exists X \; FX = X$

(ii) FIXED POINT COMBINATOR $Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$

$\forall F \; F(YF) = YF$

CONTEXT LEMMA $C[f, \vec{x}]$

$\exists F \forall \vec{x} \quad F\vec{x} = C[f, \vec{x}]$

$F := Y(\lambda f \vec{x}. C[f, \vec{x}])$

$F\vec{x} = ((\lambda f \vec{x}. C[f, \vec{x}])F)\vec{x} = (\lambda \vec{x}. C[F, \vec{x}])\vec{x} = C[F, \vec{x}]$

CHURCH NUMERALS

$c_n \equiv \lambda f x. f^n(x)$

J. B. ROSSER :

$A_+ = \lambda x y p q. xp(ypq)$

$A_* = \lambda x y z. x(yz)$

$A_{exp} = \lambda x y. yx$

$A_+ c_n c_m = \lambda pq. c_n p (c_m pq) =$

$= \lambda pq. (\lambda x. p^n(x))(p^m(q)) =$

$= \lambda pq. p^n(p^m(q)) \equiv c_{n+m}$

$$A_* \, c_n \, c_m = \lambda z.\, c_n (c_m z) =$$
$$= \lambda z. \left(\lambda f x.\, f^n(x)\right)\left(\lambda y.\, z^m(y)\right) =$$
$$= \lambda z.\, \lambda x.\, \left(\lambda y.\, z^m(y)\right)^n (x) =$$
INDUCTION
$$= \lambda z.\, \lambda x.\, \left(\lambda y.\, z^{m*n}(y)\right)(x) =$$
$$= \lambda z x.\, z^{m*n}(x) \equiv c_{m*n}$$

LEMMA: (i) $(c_n x)^m (y) = x^{n*m}(y)$

(ii) $(c_n)^m (x) = c_{(n^m)}(x) \quad m > 0$

(ii) INDUCTION

$m = 1$ ✓

$$(c_n)^{m+1}(x) = c_n\left((c_n)^m(x)\right) = c_n\left(c_{(n^m)}(x)\right) =$$
$$= \left(\lambda f y.\, f^n(y)\right)\left(c_{(n^m)}(x)\right) =$$
$$= \lambda y. \left(c_{(n^m)}(x)\right)^n (y) =$$
$$= \lambda y.\, x^{n^{m+1}}(y) =$$
$$= c_{(n^{m+1})}(x)$$

$$A_{exp} \, c_n \, c_m = c_m \, c_n = \lambda x. \, (c_n)^m (x) =$$
$$= \lambda x.\, c_{(n^m)}(x) = c_{(n^m)}$$

SINCE $\lambda z.\, c_n(z) = \lambda z. \left(\lambda f x. f^n(x)\right) z =$

$\lambda z. \lambda x.\, z^n(x) \equiv c_n$

19.05.2009 Tu
PETER ČERNO

NAIL078 LAMBDA-KALKULUS I - SKÚŠKA

5.1. $S \equiv \lambda xyz.xz(yz)$ $\quad K = \lambda xy.x$

(i) $\vdash SK : (\alpha \to \beta) \to (\alpha \to \alpha)$

$\cancel{x : \alpha \to \beta \to \gamma}$ (3)

$\cancel{y : \alpha \to \beta}$ (2)

$\cancel{z : \alpha}$ (1)

$xz : \beta \to \gamma$

$yz : \beta$

$xz(yz) : \gamma$

(1) $\lambda z.xz(yz) : \alpha \to \gamma$

(2) $\lambda yz.xz(yz) : (\alpha \to \beta) \to (\alpha \to \gamma)$

(3) $\lambda xyz.xz(yz) : (\alpha \to \beta \to \gamma) \to (\alpha \to \beta) \to (\alpha \to \gamma)$

$\cancel{x : \alpha}$ (2)

$\cancel{y : \beta}$ (1)

---

(1) $\lambda y.x : \beta \to \alpha$

(2) $\lambda xy.x : \alpha \to \beta \to \alpha$

$S \equiv \lambda xyz.xz(yz) : (\alpha \to \beta \to \alpha) \to (\alpha \to \beta) \to (\alpha \to \alpha)$

$K \equiv \lambda xy.x : \alpha \to \beta \to \alpha$

$SK : (\alpha \to \beta) \to (\alpha \to \alpha)$

(ii)

$$\frac{\begin{array}{l} K \equiv \lambda xy.x : (\alpha \to \alpha) \to \beta \to (\alpha \to \alpha) \\ I \equiv \lambda x.x : \alpha \to \alpha \end{array}}{KI : \beta \to (\alpha \to \alpha)}$$

(iii) $\nvdash SK : \alpha \to \beta \to \beta$

$S \equiv \lambda xyz.xz(yz)$
$K \equiv \lambda xy.x$

PRE SPOR PREDPOKLADAJME, ŽE
$\vdash SK : \alpha \to \beta \to \beta$

GENERATION LEMMA: $\exists \sigma$:
$\vdash S : \sigma \to (\alpha \to \beta \to \beta)$
$\vdash K : \sigma$

$\lambda xyz.xz(yz) : \sigma \to \alpha \to \beta \to \beta$

GENERATION LEMMA: $\exists \rho, \tau$:
$x : \rho \vdash \lambda yz.xz(yz) : \tau$
$\sigma \to \alpha \to \beta \to \beta \equiv \rho \to \tau$

NUTNE $\rho = \sigma$, $\tau = \alpha \to \beta \to \beta$

TJ. $x : \sigma \vdash \lambda yz.xz(yz) : \alpha \to \beta \to \beta$

GENERATON LEMMA: $\exists \rho, \tau$:
$x : \sigma, y : \rho \vdash \lambda z.xz(yz) : \tau$
$\alpha \to \beta \to \beta \equiv \rho \to \tau$

NUTNE $\rho = \alpha$, $\tau = \beta \to \beta$

TJ. $x : \sigma, y : \alpha \vdash \lambda z.xz(yz) : \beta \to \beta$

$\rightsquigarrow$ PODOBNE $x : \sigma, y : \alpha, z : \beta \vdash xz(yz) : \beta$

19.05.2009 TU
PETER ČERNO

NAIL078 LAMBDA-KALKULUS I - SKÚŠKA

OZNAČME. $\Gamma = \{x : \sigma, y : \alpha, z : \beta\}$

$\Gamma \vdash xz(yz) : \beta$

GENERATION LEMMA:

$\exists \rho : \Gamma \vdash xz : \rho \to \beta \wedge \Gamma \vdash yz : \rho$

$\Gamma \vdash yz : \rho \Rightarrow \exists \tau : \Gamma \vdash y : \tau \to \rho \wedge \Gamma \vdash z : \tau$

NUTNE $\tau \equiv \beta$ A $(\tau \to \rho) \equiv \alpha$, ČO JE SPOR ↯

(iv) $SK \equiv (\lambda xyz.\, xz(yz))(\lambda xy.\, x) \to_\beta$

$\to_\beta \lambda yz.\, (\lambda xy.\, x)\, z\, (yz) \to_\beta$

$\to_\beta \underline{\underline{\lambda yz.\, z}} \equiv K_*$

$KI \equiv (\lambda xy.\, x)(\lambda z.\, z) \to_\beta$

$\to_\beta \lambda y.\, (\lambda z.\, z) \equiv \underline{\underline{\lambda yz.\, z}} \equiv K_*$

~~$z : \alpha$~~ ~~$y : \beta$~~

$\lambda z.\, z : \alpha \to \alpha$

$\underline{\underline{\lambda yz.\, z : \beta \to \alpha \to \alpha}}$

5.2. NECH $\vdash \lambda x.xx : \sigma$

GL: $\exists \rho, \tau : \quad x:\rho \vdash xx : \tau, \quad \sigma = \rho \to \tau$

$x:\rho \vdash xx : \tau$

GL: $\exists \varphi, \mu : \quad x:\rho \vdash x : \varphi \to \mu, \quad x:\rho \vdash x : \varphi$

$\Rightarrow \rho = \varphi \to \mu \wedge \rho = \varphi$, ČO JE SPOR

PRE $KI(\lambda x.xx)$ STAČÍ POUŽIŤ VETU, ŽE KAŽDÝ PODTERM MUSÍ MAŤ TYP.

23.05.2009 Sa
PETER ČERNO

NAIL078 LAMBDA-KALKULUS I – SKÚŠKA

FUNCTIONAL PROGRAM

VARIABLE CONVENTION

APPLICATION

ABSTRACTION

(β), REDEX, CONTRACTUM

FORMAL DESCRIPTION, FV

PRINCIPAL AXIOM SCHEME + LOGICAL AXIOMS

I, K, K_*, S

EQUALITY + COMPATIBILITY RULES

FIXED POINT THEOREM, FIXED POINT COMBINATOR

$$\boxed{Y \equiv \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))}$$

$\forall F: YF = F(YF)$

CONTEXT LEMMA $C[f, \vec{x}]$

$$\boxed{F := Y(\lambda f \vec{x}.C[f, \vec{x}])}$$

$F = (\lambda f \vec{x}.C[f, \vec{x}])F \ldots$

CHURCH NUMERALS

$$\boxed{c_n = \lambda f x.f^n(x)}$$

LEMMA (ROSSER)

$$\boxed{\begin{aligned} A_+ &= \lambda xypq.xp(ypq) \\ A_* &= \lambda xyz.x(yz) \\ A_{exp} &= \lambda xy.yx \end{aligned}}$$

LEMMA:

(i) $(c_n x)^m(y) = x^{n*m}(y)$

(ii) $(c_n)^m(x) = (c_{n^m})(x)$

$m > 0$

true $= K$, false $= K_*$

ORDERED PAIR $[M, N] = \lambda z.zMN$

NUMERALS: $\ulcorner 0 \urcorner = I$, $\ulcorner n+1 \urcorner = [\text{false}, \ulcorner n \urcorner]$

$S^+ \equiv \lambda x.[\text{false}, x]$ $\quad P^- = \lambda x.x\,\text{false}$ $\quad \text{Zero} \equiv \lambda x.x\,\text{true}$

f <u>λ-DEFINABLE</u> BY F

THEOREM: ALL RECURSIVE FUNCTIONS ARE λ-DEF.

(a) BASIC FUNCTIONS U^i_n, S^+, Z

(b) OPERATIONS:

COMPOSITION

PRIMITIVE RECURSION

(REGULAR) MINIMALIZATION $H\vec{x}y$, $F := \lambda\vec{x}.H\vec{x}\ulcorner 0\urcorner$

$$\boxed{\begin{array}{l} T\,c_n = \ulcorner n\urcorner \\ \underline{T^{-1}\ulcorner n\urcorner = c_n} \\ T = \lambda x.\,x\,S^+\ulcorner 0\urcorner \\ T^{-1} = \lambda x.\ \text{if zero } x \text{ - t. } c_0 \ldots \end{array}}$$

MULTIPLE FIXED POINT THEOREM

—"— CONTEXT LEMMA

<u>CODING λ-TERMS</u> $\langle\cdot,\cdot\rangle : \mathbb{N}^2 \to \mathbb{N}$

$\#V^{(n)}$, $\#C^{(n)}$, $\#(MN)$, $\#(\lambda x.M)$,

$\ulcorner M\urcorner := \ulcorner \# M\urcorner$

<u>KLEENE</u>: $\exists$ INTERPRETER COMBINATOR E

$\forall$ <u>CLOSED</u> λ-TERMS <u>WITHOUT CONSTANTS</u>: $E\ulcorner M\urcorner = M$

IDEA: T: $\forall F$

$T\,F\,\ulcorner x\urcorner = F\ulcorner x\urcorner$

$T\,F\,\ulcorner MN\urcorner = (T\,F\,\ulcorner M\urcorner)(T\,F\,\ulcorner N\urcorner)$

$T\,F\,\ulcorner \lambda x.M\urcorner = \lambda z.(T\,(F_{[x:=z]})\,\ulcorner M\urcorner)$

WHERE

$$F_{[x:=z]}\ulcorner n\urcorner = \begin{cases} F\ulcorner n\urcorner & \text{IF } \#n \neq \#x \\ z & \text{IF } \#n = \#x \end{cases}$$

$F_{[x:=z]} = G\ulcorner x\urcorner z F$, WDE

$G \equiv \lambda xzfn.(\text{if } \mathrm{CMP}\ n\ x \text{ then } z \text{ else } (fn))$

23.05.2009 Sa
PETER ČERNO

NAIL078 LAMBDA-KALKULUS I – SKÚŠKA

BY INDUCTION ON M:

$T\,F\,\ulcorner M \urcorner = M[x_1 := F\ulcorner x_1 \urcorner, \dots][c_1 := F\ulcorner c_1 \urcorner, \dots]$

$FV(M) = \{x_1, \dots\}$, $\{c_1, \dots\}$ = CONSTANTS IN M

$E := \underline{\underline{\lambda x.\, T\,I\,x = T\,I}}$.

SECOND FIXED POINT THEOREM $\forall F \exists X\; F\ulcorner X \urcorner = X$

$AP\ulcorner M \urcorner \ulcorner N \urcorner = \ulcorner MN \urcorner$

$Num\ulcorner n \urcorner = \ulcorner \ulcorner n \urcorner \urcorner$

$$\boxed{W := \lambda x.\, F(AP\; x\; (Num\; x)) \qquad X \equiv W\ulcorner W \urcorner}$$

$X \equiv W\ulcorner W \urcorner = F(AP\,\ulcorner W \urcorner\,(Num\,\ulcorner W \urcorner)) =$
$= F\ulcorner W\ulcorner W \urcorner \urcorner = F\ulcorner X \urcorner$

<u>SCOTT THEOREM</u> $A \subseteq \Lambda$

(i) $A \neq \emptyset$, $A \neq \Lambda$
(ii) A IS CLOSED UNDER $=$

$\Rightarrow$ A IS NOT RECURSIVE

PROOF BY CONTRADICTION: SUPPOSE $\exists$ CLOSED F

$M \in A \Leftrightarrow F\ulcorner M \urcorner = \ulcorner 0 \urcorner$, $M \notin A \Leftrightarrow F\ulcorner M \urcorner = \ulcorner 1 \urcorner$

$M_0 \in A$, $M_1 \notin A$

$G := \lambda x.$ if Zero $(F\,x)$ then M_1 else M_0

$\exists M: G\ulcorner M \urcorner = M$ ↯

M IS IN / HAS A NORMAL FORM

$\Omega = (\lambda x.xx)(\lambda x.xx)$

<u>CHURCH, SCOTT</u> $NF = \{M \mid M \text{ HAS A NF}\}$ IS NOT RECURSIVE

SEMANTICS OF LAMBDA CALCULUS

(1) OPERATIONAL : REDUCTIONS & STRATEGIES
(2) DENOTATIONAL

BINARY RELATION R ON Λ :
COMPATIBLE REDUCTION CONGRUENCE

$\to_\beta$, $\twoheadrightarrow_\beta$, $=_\beta$

<u>CHURCH-ROSSER THEOREM</u> :

$$\begin{tikzcd} & M \arrow[dl, two heads, "\beta"'] \arrow[dr, two heads, "\beta"] & \\ M_1 \arrow[dr, dotted, two heads] & & M_2 \arrow[dl, dotted, two heads] \\ & N & \end{tikzcd}$$

COROLLARY : $M_1 =_\beta M_2 \Rightarrow$
$\exists N : M_1 \twoheadrightarrow_\beta N,\ M_2 \twoheadrightarrow_\beta N$

(i) IF N IS β-NF OF M $\Rightarrow$ $M \twoheadrightarrow_\beta N$
(ii) EVERY λ-TERM HAS AT MOST ONE β-NF.

TURING'S FIXED POINT COMBINATOR

$$\boxed{\Theta = AA, \quad A = \lambda xy.y(xxy)}$$

$\Theta F \twoheadrightarrow F(\Theta F)$

<u>LEFTMOST STRATEGY, LAZY STRATEGY</u>

CURRY THM. : M HAS NF N $\Rightarrow$ $M \twoheadrightarrow_\ell N$

23.05.2009 Sa
PETER ČERNO

NAIL078 LAMBDA-KALKULUS I – SKÚŠKA

DELTA RULES

- EXTENDING THE LAMBDA-CALCULUS BY "EXTERNAL" FUNCTIONS

$X \subseteq \Lambda$ SET OF CLOSED TERMS IN NORMAL FORMS (USUALLY $X \subseteq C$)

$f : X^k \to \Lambda$ EXTERNALLY DEFINED FUNCTION

(1) SPECIAL CONSTANT IS SELECTED: δ $(= \delta_f)$

(2) NEW CONTRACTION RULES: (SCHEME)

$$\delta M_1 \dots M_k \to f(M_1, \dots, M_k) \qquad M_1, \dots, M_k \in X$$

MITSCHKE THM: $\twoheadrightarrow_{B\delta}$ SATISFIES CH-R THEOREM.

THE $B\delta$-NF CAN BE FOUND BY A LEFTMOST REDUCTION.

THE PROOF OF CHURCH-ROSSER THEOREM

STRIP LEMMA

$$\begin{tikzcd}
 & M \arrow[dl, "B"'] \arrow[dr, two heads, "B"] & \\
N_1 \arrow[dr, two heads, dashed] & & N_2 \arrow[dl, two heads, dashed] \\
 & N_3 &
\end{tikzcd}$$

LET $M \to_B N_1$ BE A ONE-STEP REDUCTION RESULTING FROM CHANGING REDEX R IN M IN ITS CONTRACTUM R' IN N_1

$\rightsquigarrow$ BOOKKEEPING OF WHAT HAPPENS WITH R DURING $M \twoheadrightarrow_B N_2$

DEFINITION <u>UNDERLINING</u> $\underline{\Lambda}$, $\to_{\underline{\beta}}$, $\twoheadrightarrow_{\underline{\beta}}$

$|\cdot| : \underline{\Lambda} \to \Lambda$, $\varphi : \underline{\Lambda} \to \Lambda$ $\quad \xrightarrow{+} M, N \in \underline{\Lambda}, x \in V \Rightarrow ((\underline{\lambda} x.M)N) \in \underline{\Lambda}$

$\varphi(x) \equiv x$, $\varphi(MN) = \varphi(M)\,\varphi(N)$, $\varphi(\lambda x.M) = \lambda x.\varphi(M)$

$\varphi(\underline{(\underline{\lambda} x.M)N}) = \varphi(M)[x := \varphi(N)]$

LEMMA :

$$\begin{tikzcd}
M' \arrow[r, dashed, two heads, "\underline{\beta}"'] \arrow[d, "\shortparallel"'] & N' \arrow[d, dashed, "\shortparallel"] \\
M \arrow[r, two heads, "\beta"'] & N
\end{tikzcd}$$

LEMMA :

(i) $\varphi(M[x := N]) \equiv \varphi(M)[x := \varphi(N)]$

(ii)

$$\begin{tikzcd}
M \arrow[r, two heads, "\underline{\beta}"] \arrow[d, "\varphi"'] & N \arrow[d, "\varphi"] \\
\varphi(M) \arrow[r, dashed, two heads, "\beta"] & \varphi(N)
\end{tikzcd}$$

LEMMA :

$$\begin{tikzcd}
 & M \arrow[dl, "\shortparallel"'] \arrow[dr, "\varphi"] & \\
N \arrow[rr, dashed, two heads, "\beta"'] & & L
\end{tikzcd}$$

$M - R \equiv (\lambda x.P)Q \to N_1$

LET $M' \in \underline{\Lambda}$ BE OBTAINED FROM M BY REPLACING R BY $R' \equiv (\underline{\lambda} x.P)Q$

<u>STRIP LEMMA</u> :

$$\begin{tikzcd}
M \arrow[rr, two heads, "\beta"] \arrow[dd, "\beta"'] & & N_2 \arrow[dd, dashed, "\beta"] & \\
 & M' \arrow[ul, "\shortparallel"'] \arrow[dl, "\varphi"] \arrow[rr, dashed, two heads, "\underline{\beta}"] & & N_2' \arrow[ul, dashed, "\shortparallel"'] \arrow[dl, "\varphi"] \\
N_1 \arrow[rr, dashed, two heads, "\beta"'] & & N_3 &
\end{tikzcd}$$

24. 05. 2009 Su
PETER ČERNO

NAIL078 LAMBDA-KALKULUS I - SKÚŠKA

TYPE ASSIGNMENT

PARTIAL SPECIFICATION
PARTIAL CORRECTNESS PROOF FOR THE PROGRAM
EFFICIENCY

TYPING A LA CURRY / A LA CHURCH

$\boxed{\text{SYSTEM } \lambda\to \text{ CURRY}}$

(i) SET OF TYPES OF $\lambda\to$, $\mathbb{T} = Type(\lambda\to)$
(ii) FINITE SET OF RULES

$\alpha, \alpha', \alpha'', \ldots \in \mathbb{T}$ TYPE VARIABLES
$\sigma, \tau \in \mathbb{T} \Rightarrow (\sigma \to \tau) \in \mathbb{T}$ FUNCTION SPACE TYPES

$M:\sigma$ IS DERIVABLE FROM A BASIS Γ:
$\Gamma \vdash_{\lambda\to} M:\sigma$

$(x:\sigma) \in \Gamma \Rightarrow \Gamma \vdash x:\sigma$
$\Gamma \vdash M:(\sigma\to\tau),\ \Gamma : N:\sigma \Rightarrow \Gamma \vdash (MN):\tau$ -ELIMINATION
$\Gamma, x:\sigma \vdash M:\tau \Rightarrow \Gamma \vdash (\lambda x.M):(\sigma\to\tau)$ -INTRODUCTION

BASIC LEMMA

(i) $\Gamma' \supseteq \Gamma,\ \Gamma \vdash M:\sigma \Rightarrow \Gamma' \vdash M:\sigma$
(ii) $\Gamma \vdash M:\sigma \Rightarrow FV(M) \subseteq dom(\Gamma)$
(iii) $\Gamma \vdash M:\sigma \Rightarrow \Gamma|_{FV(M)} \vdash M:\sigma$

GENERATION LEMMA

(i) $\Gamma \vdash x : \sigma \Rightarrow x : \sigma \in \Gamma$

(ii) $\Gamma \vdash (MN) : \tau \Rightarrow \exists \sigma :$

$\Gamma \vdash M : (\sigma \to \tau)$ & $\Gamma \vdash N : \sigma$

(iii) $\Gamma \vdash (\lambda x. M) : \rho \Rightarrow \exists \sigma, \tau :$

$\Gamma, x : \sigma : M : \tau$ & $\rho = (\sigma \to \tau)$

TYPABILITY OF SUBTERMS

SUBSTITUTION LEMMA

(i) $\Gamma : M : \sigma \Rightarrow \Gamma[\alpha := \tau] \vdash M : \sigma[\alpha := \tau]$

(ii) $\Gamma, x : \sigma \vdash M : \tau$ & $\Gamma \vdash N : \sigma \Rightarrow \Gamma \vdash M[x := N] : \tau$

SUBJECT REDUCTION THEOREM

$M \twoheadrightarrow_\beta M' \Rightarrow \Gamma \vdash M : \sigma \Rightarrow \Gamma \vdash M' : \sigma$

THE SET OF TYPEABLE TERMS IS NOT CLOSED UNDER EXPANSION

$\vdash I : (\sigma \to \sigma)$, $\nvdash K I (\lambda x. xx) : (\sigma \to \sigma)$

VAN BAKEL $\exists M, M' \in \Lambda$, $\sigma, \sigma' \in T$, $M \twoheadrightarrow_\beta M'$

$\vdash M : \sigma$, $\vdash M' : \sigma'$, $\nvdash M : \sigma'$

$M \equiv SK : (\beta \to \alpha) \to (\beta \to \beta)$

$M' \equiv K_* : \alpha \to (\beta \to \beta)$

$SK \twoheadrightarrow_\beta K_*$

$\nvdash SK : \alpha \to (\beta \to \beta)$

24.05.2009 Su
PETER ČERNO

NAIL078 LAMBDA-KALKULUS I - SKUŠKA

$\boxed{\text{SYSTEM } \lambda\text{-CHURCH}}$

$\mathbb{T}$-ANNOTATED λ-TERMS (PSEUDOTERMS)

$$\Lambda_T = V \mid (\Lambda_T \Lambda_T) \mid (\lambda x : \mathbb{T} . \Lambda_T)$$

$\mathbb{T} = \mathrm{Type}(\lambda\to)$ AS FOR $\lambda\to$-CURRY

$(x:\sigma) \in \Gamma \Rightarrow \Gamma \vdash x : \sigma$

$\Gamma \vdash M : (\sigma \to \tau)$ & $\Gamma \vdash N : \sigma \Rightarrow \Gamma \vdash (MN) : \tau$

$\Gamma, x : \sigma \vdash M : \tau \Rightarrow \Gamma \vdash (\lambda x : \sigma . M) : (\sigma \to \tau)$

<u>LEGAL $\lambda\to$-TERMS</u> $\Lambda(\lambda\to)$

$= \{ M \in \Lambda_T \mid \exists \Gamma, \sigma : \Gamma \vdash M : \sigma \}$

<u>BASIC LEMMA</u>

<u>GENERATION LEMMA</u>

<u>TYPABILITY OF SUBTERMS</u>

<u>SUBSTITUTION LEMMA</u>

(i) $\Gamma \vdash M : \sigma \Rightarrow \Gamma[\alpha := \tau] \vdash M[\alpha := \tau] : \sigma[\alpha := \tau]$

(ii) ...

<u>SUBJECT REDUCTION THEOREM</u>

<u>LEMMA ON UNIQUENESS OF TYPES</u> (ONLY FOR λ-CHURCH)

(i) $\Gamma \vdash M : \sigma$ & $\Gamma \vdash M : \sigma' \Rightarrow \sigma \equiv \sigma'$

(ii) $\Gamma \vdash M : \sigma$ & $\Gamma \vdash M' : \sigma'$ & $M =_\beta M' \Rightarrow \sigma = \sigma'$

RELATING THE CURRY AND CHURCH SYSTEMS

$|\cdot| : \Lambda_T \to \Lambda$ (ERASES ANNOTATIONS)

$|x| \equiv x, \quad |MN| \equiv |M||N|, \quad |\lambda x{:}\sigma . M| \equiv \lambda x . |M|$

THEOREM :

(i) PROJECTION $\quad M \in \Lambda_T$

$\Gamma \vdash_{CHURCH} M : \sigma \Rightarrow \Gamma \vdash_{CURRY} |M| : \sigma$

(ii) LIFTING $\quad M \in \Lambda$

$\Gamma \vdash_{CURRY} M : \sigma \Rightarrow \exists M' \in \Lambda_T : \Gamma \vdash_{CHURCH} M' : \sigma \ \& \ |M'| \equiv M$

COROLLARY: $\sigma \in \mathbb{T}$

σ IS INHABITED IN λ-CURRY $\Leftrightarrow$
σ IS INHABITED IN λ-CHURCH.

CURRY

STRONG NORMALIZATION PROPERTY

IF M IS A TYPEABLE TERM $\Rightarrow$ ALL REDUCTIONS STARTING FROM M ARE FINITE.

DECIDABILITY OF TYPE ASSIGNMENT

$M : \sigma$? TYPE CHECKING

$M : ?$ TYPABILITY

$? : \sigma$ INHABITATION