Poznámky z přednášek
Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

Kombinatorika a grafy II
BONUS [1]

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[1] v současnosti Kombinatorika a grafy II a III

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Anotace: Přehledová přednáška o klasických výsledcích v kombinatorice a teorii grafů. Přednáška je určena primárně pro studenty magisterského stupně, popř. studenty prvního ročníku doktorského studia.

Sylabus:

1. Klasické výsledky z teorie grafů (Brooksova a Vizingova věta, Tutteho věta a její algoritmický důkaz).

2. Klasické výsledky z exremální kombinatoriky (Turanova věta, Ramseyova teorie), příklady netradičních důkazových technik (pravděpodobnostní metoda).

3. Nové trendy v teorii grafů (vybíravost grafu, Szemeredi Regularity Lemma, Removal Lemma), pokročilé výsledky z extremální kombinatoriky (Hales- Jewettova věta, Gallai-Wittova věta).

4. Přednáška probíhá v angličtině, cvičení česky.

Literatura:

1. R. Diestel, Graph theory, 3rd edition, Springer, 2005.

2. S. Jukna, Extremal combinatorics with application in computer science, Springer, 2001.

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DMI012 KOMBINATORIKA A GRAFY II - SKÚŠKA

PROPOSITION 1.2.2: EVERY GRAPH WITH AT LEAST ONE EDGE HAS A SUBGRAPH H WITH $\delta(H) > \varepsilon(H) \geq \varepsilon(G)$.

$\delta(G) = \min\{d(v) \mid v \in V\}$, $\Delta(G) = \max\{d(v) \mid v \in V\}$,
$d(G) = \frac{1}{|V|} \sum_{v \in V} d(v)$, $\varepsilon(G) = |E|/|V|$

HINT: DELETE VERTICES WITH $d(v) \leq \varepsilon$

$\delta(G) \leq d(G) \leq \Delta(G)$, $\varepsilon(G) = \frac{1}{2} d(G)$

GIRTH $g(G)$ = MINIMUM LENGTH OF CYCLE IN G (∞)
CIRCUMFERENCE = MAXIMUM —ıı— (0)

IF A GRAPH HAS LARGE MINIMUM DEGREE, IT CONTAINS LONG PATHS AND CYCLES.

PROPOSITION 1.3.1: EVERY GRAPH G CONTAINS A PATH OF LENGTH $\delta(G)$ AND A CYCLE OF LENGTH AT LEAST $\delta(G)+1$ (PROVIDED THAT $\delta(G) \geq 2$). HINT: TAKE THE LONGEST PATH

MINIMUM DEGREE AND GIRTH ARE NOT RELATED

$\mathrm{diam}(G) = \max_{x,y \in V(G)} d_G(x,y)$, $\mathrm{rad}(G) = \min_{x \in V(G)} \max_{y \in V(G)} d_G(x,y)$

PROPOSITION 1.3.2: EVERY GRAPH G CONTAINING A CYCLE SATISFIES $g(G) \leq 2\,\mathrm{diam}(G) + 1$. HINT: WHAT IF $g(G) \geq 2\,\mathrm{diam}(G)+2$?

$\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\,\mathrm{rad}(G)$ HINT: $\forall$ x-y PATHS: $d_G(x,y) \leq 2\,\mathrm{rad}(G)$

DIAMETER (RADIUS) ARE NOT DIRECTLY RELATED TO MIN./AVG. DEGREE

PROPOSITION 1.3.3: A GRAPH OF RADIUS $\leq k$ AND MAX. DEG. $\leq d$ HAS NO MORE THAN $1 + kd^k$ VERTICES.
HINT: LET D_i DENOTE SET OF VERTICES AT DIST. i FROM CEN. VERTEX z

PROPOSITION 1.4.1: THE VERTICES OF A CONNECTED GRAPH G CAN ALWAYS BE ENUMERATED $v_1, \dots, v_n$: $G_i = G[v_1, \dots, v_i]$ IS CONN.

G IS k-CONNECTED IF $|G| > k$ AND $\forall X \subseteq V$ $|X| < k$, $G - X$ IS CONN.
CONNECTIVITY $\kappa(G)$ = GREATEST SUCH k

G IS ℓ-EDGE-CONNECTED IF $|G| > 1$ AND $\forall F \subseteq E$ $|F| < \ell$, $G - F$ IS C.
EDGE-CONNECTIVITY $\lambda(G)$

$\kappa(G) \leq \lambda(G) \leq \delta(G)$

HINT: $\lambda(G)-1 \leq \lambda(G-e) \leq \lambda(G)$

$\kappa(G)-1 \leq \kappa(G-e) \leq \kappa(G)$ IE FOR $H = G-e$: $\kappa(H+e) \leq \kappa(H)+1$

INDUCTION ON $\|G\|$.

FORD-FULKERSON: $\forall G\ \forall t \in \mathbb{N}\ \lambda(G) \geq t \Leftrightarrow \forall u \neq v \in V(G)$ $\exists$ AT LEAST t EDGE-DISJOINT u-v PATHS

MENGER: $\forall G\ \forall t \in \mathbb{N}\ \kappa(G) \geq t \Leftrightarrow \forall u \neq v \in V(G)$ $\exists$ AT LEAST t INDEPENDENT u-v PATHS

THEOREM 1.4.2 (MADER): EVERY GRAPH OF AVERAGE DEGREE AT LEAST $4k$ HAS A k-CONNECTED SUBGRAPH.

COROLLARY 1.5.4: IF T IS A TREE AND G IS ANY GRAPH S.T. $\delta(G) \geq |T|-1$, THEN G HAS A SUBGRAPH ISOMORPHIC TO T.

PROPOSITION 1.5.5: EVERY CONNECTED GRAPH CONTAINS A NORMAL SPANNING TREE, WITH ANY SPECIFIED VERTEX ROOT r.

PROPOSITION 1.7.1: G IS AN $MX \Leftrightarrow X$ CAN BE OBTAINED FROM G BY A SERIES OF EDGE CONTRACTIONS.

THEOREM 1.8.1 (EULER): A CONNECTED GRAPH IS EULERIAN IF AND ONLY IF EVERY VERTEX HAS EVEN DEGREE.

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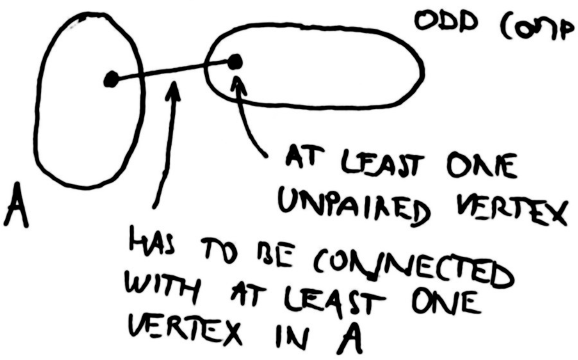
MATCHING $M \subseteq E(G)$

MAXIMAL, MAXIMUM, PERFECT MATCHING

ALTERNATING PATH
AUGUMENTING PATH

<u>TUTTE THEOREM</u> : GRAPH G HAS A PERFECT MATCHING
$\Leftrightarrow \forall A \subseteq V(G) : C_{odd}(G - A) \leq |A|$.

PROOF: $\Rightarrow$)



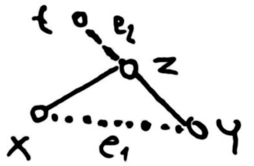
INDUCTION ON $\|G\|$ FROM $\binom{n}{2}$ TO 0.

$\Leftarrow$) IF $G = K_n \Rightarrow C_{odd}(G - \emptyset) = 0 \Rightarrow \exists$ PERFECT MATCHING

IF $G \neq K_n$, $|G| = n$, LET $W := \{v \in V(G) : deg_G(v) = n-1\}$

IF $G - W$ HAS COMPLETE COMPONENTS $\Rightarrow \exists$ MATCHING

OTHERWISE $\exists x, y, z \in V(G-W)$ $z \notin W \Rightarrow \exists t,\ zt \notin E(G-W)$



M_1 BE A PERFECT MATCHING IN $G + e_1$
M_2 -||- IN $G + e_2$

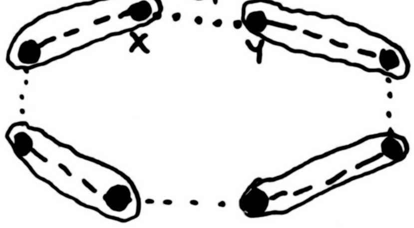
<u>ADDING AN EDGE PRESERVES TUTTE CONDITION</u>
DOES NOT ADD NEW ODD COMP.

SUPPOSE $e_1 \in M_1$, $e_2 \in M_2$.

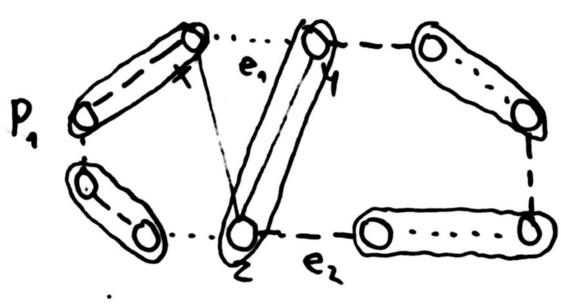
LET $\tilde{M} := M_1 \div M_2$ $\quad deg_{\tilde{M}}\, u = \begin{cases} 0 \\ 2 \end{cases}$ EVEN CYCLES

APPARENTLY $e_1, e_2 \in \tilde{M}$, C BE A CYCLE CONTAINING e_1

a) $e_2 \notin C$



b) $e_2 \in C$ WLOG



MATCHING M IS MAXIMUM $\iff$ THERE IS NO AUGUMENTING (ALTERNATING WITH FREE ENDPOINTS) PATH

$\iff$ $|M'| > |M|$ $\quad \tilde{M} = M \div M'$ CONTAINS AN ODD PATH

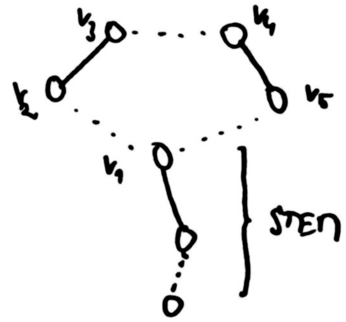
LET M BE A MATCHING $C_{2k+1} = \{v_1, \dots, v_{2k+1}\}$ IS BLOSSOM $\overset{def}{\iff}$

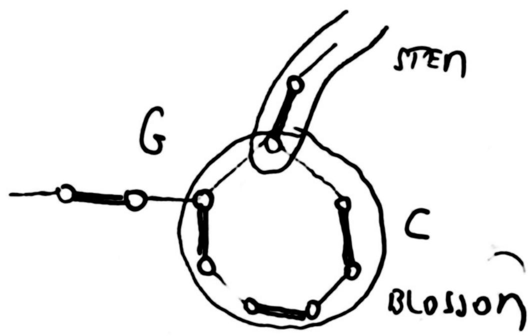
(1) v_1 IS FREE IN $M \cap E(C_{2k+1})$

(2) $\{v_{2i+1}, v_{2i+2}\} \notin M$

(3) $\{v_{2i}, v_{2i+1}\} \in M$

(4) $\exists$ STEM



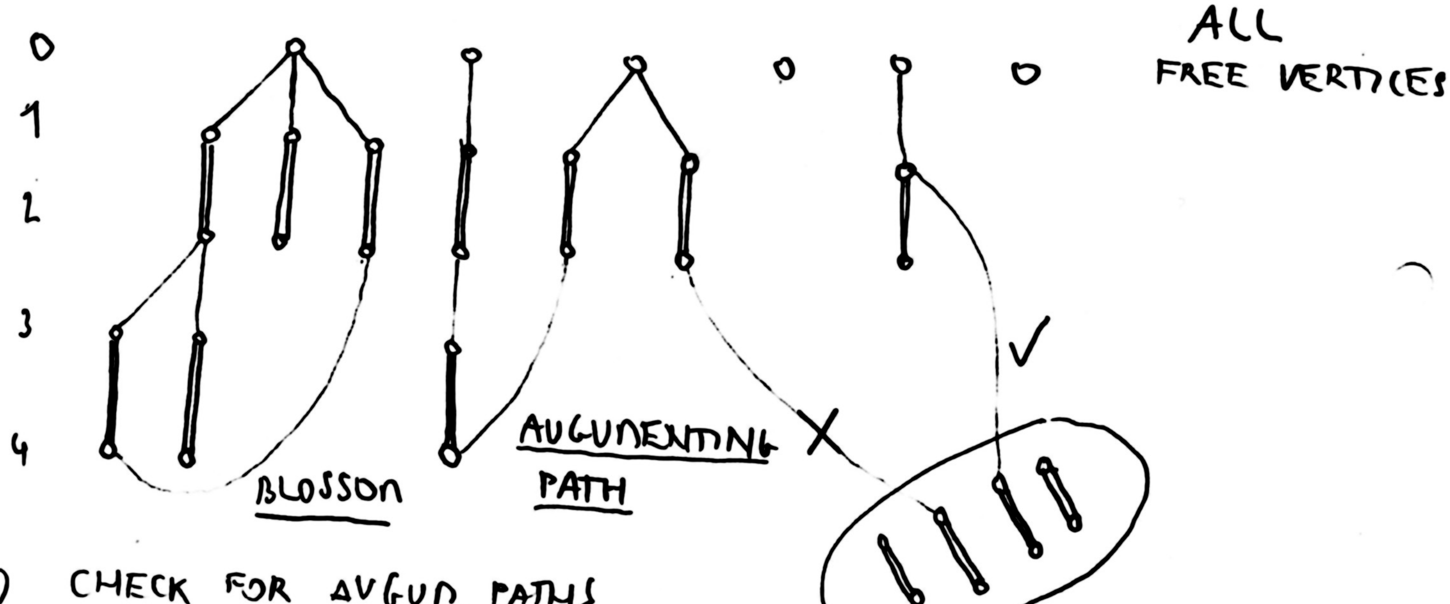


M IS MAXIMUM IN G $\iff$ $M \setminus E(C)$ IS MAXIMUM IN G.C.

EDMONDS ALGORITHM

INPUT: G, M $\qquad$ OUTPUT: M'

1) EDMONDS FOREST



2) CHECK FOR AUGUM. PATHS

3) CHECK FOR BLOSSOM → RECURSION

4) $M' := M$

$O(n(n+m))$

MAXIMUM MATCHING IN $O(n^2(n+m))$.

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DM1012 KOMBINATORIKA A GRAFY II – SKÚŠKA

LET $G=(V,E)$ BE A FIXED BIPARTITE GRAPH WITH BIPARTITION $\{A,B\}$.

<u>THEOREM</u> 2.1.1 (KÖNIG): THE MAXIMUM CARDINALITY OF A MATCHING IN G IS EQUAL TO MINIMUM CARDINALITY OF A VERTEX COVER.

<u>THEOREM</u> 2.1.2 (HALL): G CONTAINS A MATCHING OF A $\Leftrightarrow$ $|N(S)| \geq |S|$ $\forall S \subseteq A$. (MARRIAGE THEOREM) ONC

<u>COROLLARY</u> 2.1.3 : IF $|N(S)| \geq |S|-d$ $\forall S \subseteq A$, FIXED $d \in \mathbb{N}$
$\Rightarrow$ G CONTAINS A MATCHING OF CARDINALITY $|A|-d$.

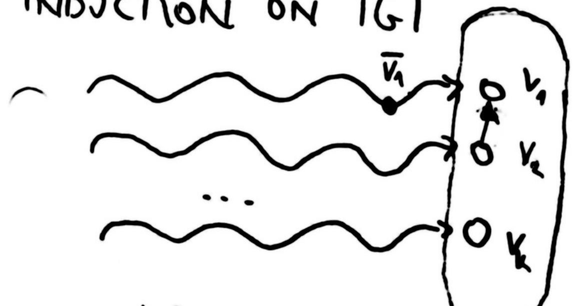
<u>COROLLARY</u> 2.1.4 : IF G IS k-REGULAR $(k \geq 1)$, THEN G HAS A 1-FACTOR.

$\alpha(G)$ MAXIMUM CARDINALITY OF AN INDEPENDENT SET OF VERTICES IN G.

<u>THEOREM</u> 2.3.1 (GALLAI & MILGRAM): EVERY DIRECTED GRAPH G HAS A PATH COVER BY AT MOST $\alpha(G)$ PATHS.

HINT: IF $P_1, \dots, P_k$ IS A PATH COVER OF G, $k > \alpha(G)$
$\Rightarrow \exists$ PATH COVER $P_1', \dots, P_r'$ S.T. $\{ter(P_1'), \dots, ter(P_r')\} \subseteq \{\underset{v_1}{ter(P_1)}, \dots, \underset{v_k}{ter(P_k)}\}$

INDUCTION ON $|G|$



A IS NOT INDEPENDENT

WLOG $v_2 \to v_1$, $\bar{v}_1$

$\exists$ PATH COVER $\bar{P}_1, \dots, \bar{P}_{\bar{k}}$ S.T. $\bar{k} < k$
$\{ter(\bar{P}_1), \dots, ter(\bar{P}_{\bar{k}})\} \subseteq \{ter(P_1), \dots, ter(P_k)\}$

(i) $ter(\bar{P}_i) = \bar{v}_1$
(ii) $ter(\bar{P}_i) = v_2$
(iii) $\bar{k} \leq k-2$

$\alpha(G - v_1) \leq \alpha(G)$

<u>COROLLARY</u> 2.3.2 (DILWORTH): IN EVERY FINITE PARTIALLY ORDERED SET $(P, \leq)$, THE MINIMUM NUMBER OF CHAINS COVERING P IS EQUAL TO MAXIMUM CARDINALITY OF AN ANTICHAIN IN P.

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DMI012 KOMBINATORIKA A GRAFY II - SKÚŠKA

THEOREM 3.5.1 (TUTTE, NASH-WILLIAMS)

A MULTIGRAPH CONTAINS k EDGE-DISJOINT SPANNING TREES IF AND ONLY IF FOR EVERY PARTITION P OF ITS VERTEX SET IT HAS AT LEAST $k(|P|-1)$ CROSS-EDGES.

LET $G=(V,E)$ AND $k\in\mathbb{N}$ BE GIVEN

$\mathcal{F}=\{F=(F_1,\dots,F_k)$ EDGE-DISJ. SPANNING FORESTS, MAX. TOT. NUMBER OF EDGES $\|F\|$ IN $G\}$

$E[F]=E(F_1)\cup\dots\cup E(F_k)$

EDGE REPLACEMENT $e\in E\setminus E[F]$ $\quad F\to F'$ $\quad (F_i'=F_i+e-e')$

$F'\notin\mathcal{F}$

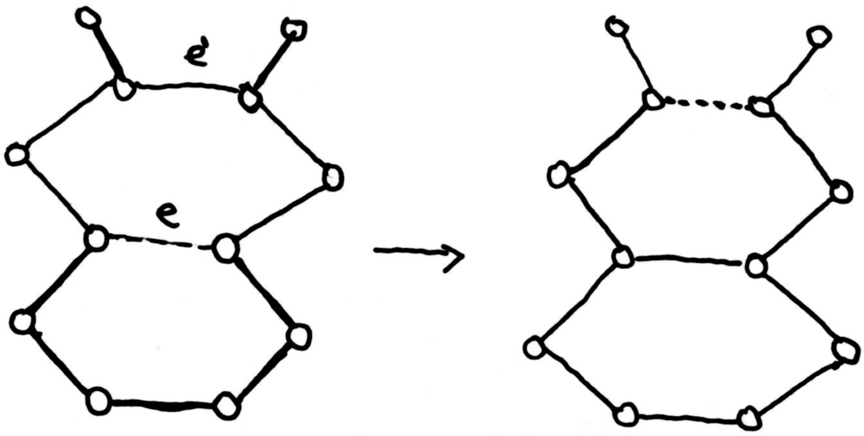
IF WE FIX $F^0=(F_1^0,\dots,F_k^0)\in\mathcal{F}$

$\Rightarrow \mathcal{F}^0:=$ SET OF ALL k-TUPLES IN $\mathcal{F}$ THAT CAN BE OBTAINED FROM F^0 BY A SERIES OF EDGE REPLACEMENTS

$E^0:=\bigcup_{F\in\mathcal{F}^0}(E\setminus E[F])$

$G^0:=(V,E^0)$

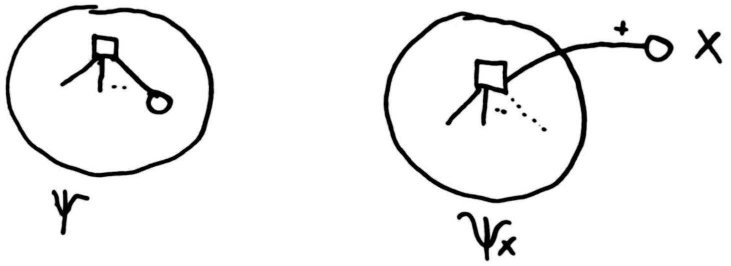
!!! ???



$(k, \leq s)$ - FORMULA

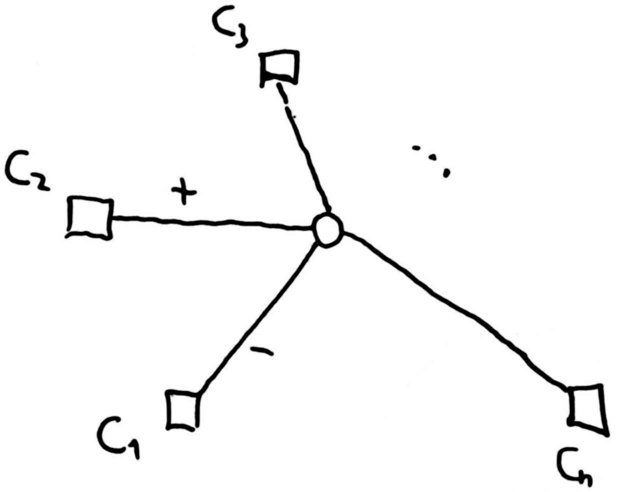
IF THERE EXISTS A NONSATISFIABLE $(k, \leq s)$ - FORMULA
$\Rightarrow$ $(k, \leq s)$ - SAT IS NP - COMPLETE

ψ IS MINIMAL NONSATISFIABLE $(k, \leq s)$ - FORMULA
$\Rightarrow \forall C_i \in \psi : \psi \setminus C_i$ IS SATISFIABLE

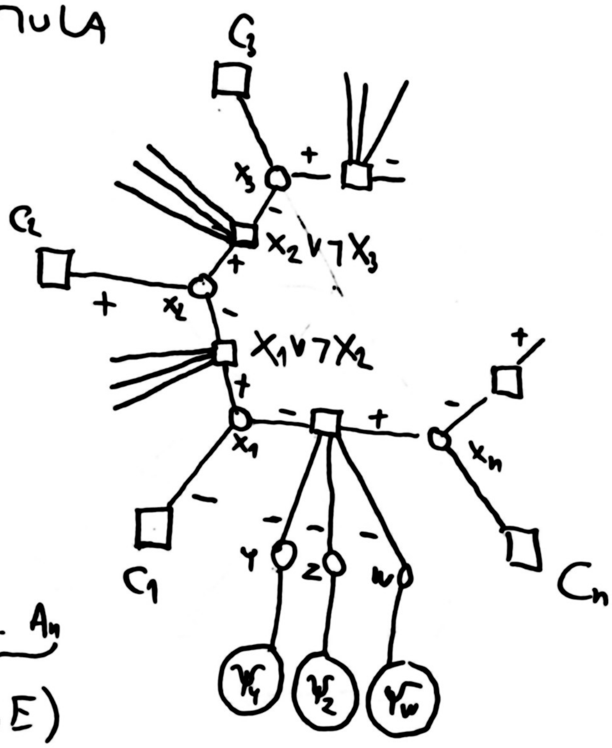


ψ_x IS SATISFIABLE, x IS TRUE IN ALL SATISFYING ASSIGNM.

LET Φ BE A $(k, \leq \infty)$ - FORMULA



$\Rightarrow$

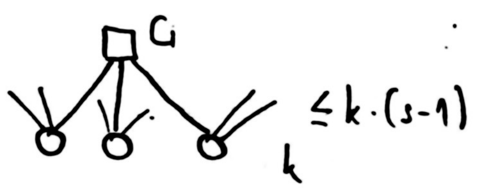


$\underbrace{A_1 \ldots A_n}$

DEPENDENCY GRAPH $G = (V, E)$

IF MAX. DEGREE OF G IS d, $P(A_i) \leq p$, $pd \leq \frac{1}{4}$
$\Rightarrow P(\bigcap \bar{A}_i) > 0$

$(k; \leq \frac{2^{k-2}}{k})$ IS SATISFIABLE



$d \leq k(s-1)$

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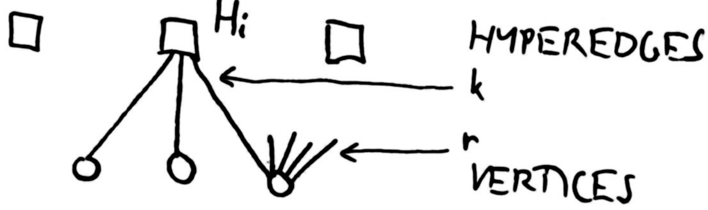
DMI012 KOMBINATORIKA A GRAFY II – SKÚŠKA

1. FOR $k \geq 3$ WE SHOW THAT EVERY k-UNIFORM r-REGULAR HYPERGRAPH, $r \leq \frac{2^{k-3}}{k}$, HAS A BICOLORING

ASSUME A RANDOM COLORING $\varphi : V \to \{0,1\}$ OF VERTICES $P(\varphi(v)=0) = P(\varphi(v)=1) = \frac{1}{2}$



BAD EVENT FOR A HYPEREDGE H_i IS ALL ITS VERTICES ARE OF THE SAME COLOR

$$p(A_i) = \frac{2}{2^k} = \frac{1}{2^{k-1}}$$

APPARENTLY $d \leq k \cdot (r-1)$

$$\Rightarrow \quad p\,d < \frac{1}{2^{k-1}} \cdot k \cdot \frac{2^{k-3}}{k} = \frac{1}{4}$$

$$\Rightarrow P\left(\bigcap \bar{A}_i\right) > 0$$

$$\Rightarrow r_0(k) \geq \frac{2^{k-3}}{k}$$

2. IF $k \geq 3$. LET $|V| = n = 2k$, ALL HYPEREDGES $\binom{V}{k}$
IF $\varphi : V \to \{0,1\}$ IS A BICOLORING,
WLOG THERE IS $\geq$ 0 COLOURED VERTICES $\to$ AT LEAST k.
$\Rightarrow$ WE HAVE FOUND UNICOLOURED HYPEREDGE.

$$\Rightarrow r_0(k) < \binom{2k-1}{k-1} < \binom{2k}{k} < 4^k$$

IF G IS k-UNIF. r-REG. HYPERGRAPH
FOR EACH HYPEREDGE $\{x_1, \dots, x_k\}$ CONSIDER
$(x_1 \vee \dots \vee x_k) \wedge (\neg x_1 \vee \dots \vee \neg x_k)$
$\leadsto$ WE GET $(k, \leq 2r)$ FORMULA ψ
WHICH IS SATISFIABLE FOR $2r \leq \frac{2^{k-2}}{k}$ IE $r \leq \frac{2^{k-3}}{k}$.

$D = h(r)$

THEOREM 3.6.1 (MADER) : $\forall r \exists D \forall G$ WITH $d(G) \geq D$
G CONTAINS A SUBDIVISION OF K_r.

INDUCTION ON $m = r, \dots, \binom{r}{2}$ THAT $\forall G$ WITH $d(G) \geq 2^m$
CONTAINS A SUBDIVISION OF A SIMPLE GRAPH WITH
r VERTICES AND m EDGES. $= TX$, $|X| = r$, $\|X\| = m$

1) $m = r$, $d(G) \geq 2^r \Rightarrow \exists$ SUBGRAPH $H \subseteq G$ SUCH THAT
$\delta(H) > \varepsilon(H) \geq \varepsilon(G) = \frac{1}{2} d(G) = 2^{r-1}$
APPARENTLY $2^{r-1} \geq r \Rightarrow \delta(H) \geq r \Rightarrow \exists$ CYCLE $C_{\delta(H)+1} \subseteq H$.

2) INDUCTION STEP $m > r$, G BE GIVEN, $d(G) \geq 2^m \Rightarrow \varepsilon(G) \geq 2^{m-1}$
WE MAY ASSUME, THAT G IS CONNECTED

LET U BE MAXIMAL CONNECTED SET OF VERTICES IN G
SUCH THAT $\varepsilon(G/U) > 2^{m-1}$ (IT EXISTS, CONSIDER $U = \{v\}$)

LET $H := G[N(U)]$ H DOES NOT CONTAIN $v : d_H(v) < 2^{m-1}$
$\Rightarrow d(H) \geq \delta(H) \geq 2^{m-1} \Rightarrow$ (BY INDUCTION) H CONTAINS
A TY WITH $|Y| = r$, $\|Y\| = m-1$

LET x, y BE TWO BRANCH VERTICES OF THIS TY, THAT
ARE NONADJACENT IN Y. SINCE $x, y \in N(U)$,
U IS CONNECTED IN G, G CONTAINS AN x-y PATH
WHOSE INNER VERTICES LIE IN U. $\Rightarrow TX$.

k-LINKED GRAPH $\Rightarrow$ k-CONNECTED

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$A, B \subseteq V$, $X \subseteq V \cup E$

X SEPARATES THE SETS A AND B IN $G \Leftrightarrow$ EVERY A-B PATH IN G CONTAINS A VERTEX OR AN EDGE FROM X.

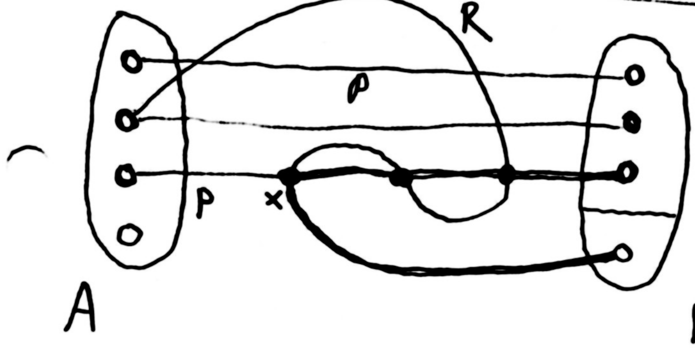
THEOREM 3.3.1 (MENGER) LET $G=(V,E)$ BE A GRAPH AND $A, B \subseteq V$. THE MINIMUM NUMBER OF VERTICES SEPARATING A FROM B IN G IS EQUAL TO THE MAXIMUM NUMBER OF DISJOINT A-B PATHS.

HINT

LET k DENOTE THE MINIMUM NUMBER OF VERTICES SEP. A FROM B IN G

WE PROVE: IF $\mathcal{P}$ IS ANY SET OF FEWER THAN k DISJOINT A-B PATHS IN $G \Rightarrow$ THERE IS A SET $\mathcal{Q}$ OF $|\mathcal{P}|+1$ DISJOINT A-B PATHS WHOSE SET OF ENDS INCLUDES THE SET OF ENDS OF THE PATHS IN $\mathcal{P}$.

G, A FIXED, INDUCTION ON $|G-B|$



LET R BE ANY A-B PATH THAT ($<k$) VERTICES THAT LIE ON A PATH IN $\mathcal{P}$

IF R DISJOINT $\Rightarrow$ OK, OTHERWISE

$B' := B \cup V(xP \cup xR)$, $\mathcal{P}' := (\mathcal{P} - \{P\}) \cup \{Px\}$

$|\mathcal{P}'| = |\mathcal{P}|$, $\underbrace{k(G,A,B') \geq k(G,A,B)}$ $\Rightarrow$ INDUCTION: $\mathcal{Q}'$ OF $|\mathcal{P}|+1$ DISJOINT A-B' PATHS WITH THE SET OF ENDS ...

IF X SEPARATES A FROM B', $\Rightarrow$ EVERY A-B PATH CAN BE SHORTENED TO A-B' PATH $\Rightarrow$ INTERSECTS X

TWO CASES: $y \notin \mathring{x}P$, $y \in \mathring{x}P$

APPLIED TO A BIPARTITE GRAPH, MENGER'S THEOREM SPECIALIZES TO THE ASSERTION OF KÖNIG'S THEOREM 2.1.1.

COROLLARY 3.3.4 LET a, b BE TWO DISTINCT VERTICES OF G

(i) IF $ab \notin E \Rightarrow$ MINIMUM NUMBER OF VERTICES $\neq a, b$ SEPARATING a FROM b = MAXIMUM NUMBER OF INDEPENDENT a-b PATHS IN G

(ii) THE MINIMUM NUMBER OF EDGES SEPARATING a FROM b = MAXIMUM NUMBER OF EDGE DISJOINT a-b PATHS IN G

HINT: (i) MENGER TO $A := N(a)$, $B := N(b)$
(ii) MENGER TO $A := E(a)$, $B := E(b)$ IN LINE GRAPH OF G

THEOREM 3.3.5 (GLOBAL VERSION OF MENGER' THEOREM)

(i) A GRAPH IS k-CONNECTED $\Leftrightarrow$ IT CONTAINS k INDEPENDENT PATHS BETWEEN ANY TWO VERTICES

(ii) A GRAPH IS k-EDGE-CONNECTED $\Leftrightarrow$ IT CONTAINS k EDGE-DISJOINT PATHS BETWEEN ANY TWO VERTICES

THEOREM 3.6.2 (JUNG, LARMAN & MANI): $\forall k \in \mathbb{N} \; \exists K = f(k)$
$\forall$ K-CONNECTED GRAPH IS k-LINKED.

HINT: $\boxed{f(k) = h(3k) + 2k}$ (h FROM 3.6.1)

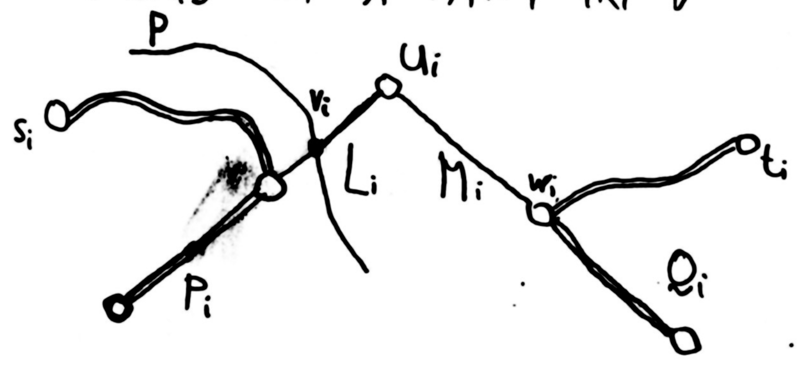
$d(G) \geq h(3k) \Rightarrow \exists K = TK^{?} \subseteq G$, $U :=$ BRANCH VERTICES OF K

LET $s_1, \dots, s_k, t_1, \dots, t_k \in V(G)$ BE GIVEN

MENGER $(A := \{s_1, \dots, s_k, t_1, \dots, t_k\}$ $B := U) \Rightarrow$ DISJOINT PATHS P_i STARTS IN s_i, Q_i IN t_i, WITH THE ENDS IN U.

LET THE SET $\mathcal{P}$ OF THESE PATHS: TOTAL NUMBER OF EDGES OUTSIDE $E(K)$ IS AS SMALL AS POSSIBLE

LET $u_1, \dots, u_k$ BE THOSE VERTICES IN U THAT ARE NOT AN END OF A PATH IN $\mathcal{P}$



$L_i :=$ U-PATH FROM u_i TO THE END OF P_i

$v_i :=$ FIRST VERTEX OF L_i ON ANY PATH $P \in \mathcal{P}$

P HAS NO MORE EDGES THAN $P v_i L_i u_i \Rightarrow v_i P = v_i L_i \Rightarrow P = P_i$

CONSIDER PATHS $s_i P_i v_i L_i u_i M_i w_i Q_i t_i$ $\square$

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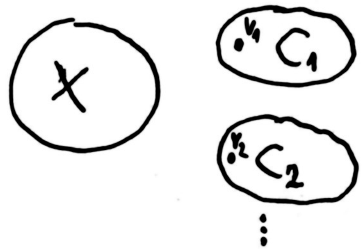
1. LET $G=(V,E)$ BE A GRAPH $|G|\geq k+1$.
G IS k-CONNECTED $\Leftrightarrow$ $\forall$ SETS $S,T\subseteq V(G)$ $|S|=|T|=k$
THERE ARE k DISJOINT S-T PATHS

$\Rightarrow)$ IF THE MAXIMUM NUMBER OF DISJOINT S-T PATHS IS $k'<k$ $\Rightarrow$ (MENGER) $\exists X\subseteq V(G)$ $|X|=k'$ SEPARATING S FROM T $\Rightarrow$ $G-X$ IS DISJOINT.

$\Leftarrow)$ SUPPOSE THAT $\exists X\subseteq V(G)$ $|X|=k'<k$: $G-X$ IS DISCONNECTED

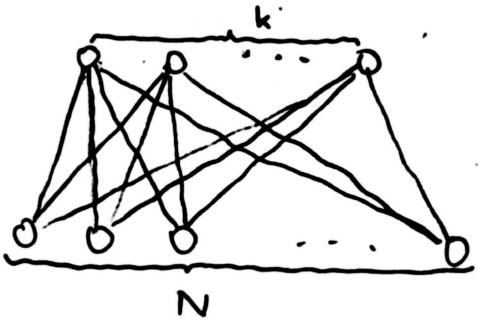


LET $U\subseteq V(G)$ SUCH THAT
$X\subseteq U$, $|U|=k-1$
$S:=U\cup\{v_1\}$, $T:=U\cup\{v_2\}$

THERE ARE k DISJOINT S-T PATHS. k-1 OF THEM ARE TRIVIAL, ONE IS v_1-v_2 PATH
SINCE $v_1\in C_1$, $v_2\in C_2$ THIS PATH HAS AN INNER VERTEX IN X. ↯

2. $k\geq 2$. k-CONNECTED GRAPH $G=(V,E)$ $|G|\geq 2k$
HAS A CYCLE C : $|C|\geq 2k$.

LET US TAKE THE LONGEST CYCLE C IN G, $|C|=\ell<2k$
CHOOSE $v\in G-C$, $A:=N(v)$, $B:=V(C)$
APPARENTLY $|A|\geq k$, $|B|\geq k+1$
MENGER : THERE ARE k DISJOINT A-B PATHS
$\Rightarrow$ k DISJOINT (EXCEPT v) v-C PATHS
WITH ENDPOINTS $e_1,\dots,e_k$ $\Rightarrow$ AT LEAST 2 ENDPOINTS ARE NEIGHBOURS ON C.



$K_{k,N}$ WHERE N IS BIG IS k CONNECTED WITH NO LONGER CYCLES THAN 2k.

SUNFLOWER LEMMA (ERDŐS - RADO) : LET $\mathcal{F}$ BE A COLLECTION OF s-ELEMENT SETS. IF $|\mathcal{F}| > s!\,(k-1)^s$, THEN $\mathcal{F}$ CONTAINS A SUNFLOWER $Y_1, \dots, Y_k$.

INDUCTION ON s

$s > 1$: CONSIDER THE LARGEST SUBFAMILY OF $\mathcal{F}$ OF MUTUALLY DISJOINT SETS $A_1, \dots, A_t$

$t < k \Rightarrow X = \bigcup A_i$ $\quad |X| \leq (k-1)s$ $\quad \exists x \in X$ CONTAINED IN MORE THAN $s!\,(k-1)^s / s(k-1)$ SETS OF $\mathcal{F}$.

THEOREM (ERDŐS - KO - RADO) : LET $\mathcal{F}$ BE A FAMILY OF k-ELEMENT SETS AND SUBSETS OF AN n-ELEMENT SET, $n \geq 2k$. IF EVERY TWO SETS IN $\mathcal{F}$ HAVE A NONEMPTY INTERSECTION, THEN $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

$\pi \in S_n$, $Y \in \mathcal{F}$ IS π-NICE $\Leftrightarrow$ $Y = \{\pi(i), \dots, \pi(i+k-1)\}$ FOR SOME $i \in \{1, \dots, n\}$

THE NUMBER OF SETS CONTAINED IN $\mathcal{F}$ THAT ARE π-NICE IS AT MOST k $\quad \forall \pi \in S_n$

$$\#(\pi, Y), \quad \underset{Y}{|\mathcal{F}|} \cdot \underset{i}{n} \cdot \underbrace{k!\,(n-k)!}_{\pi} = \#\text{PAIRS} \leq \underset{\pi}{n!}\ \underset{\pi\text{ NICE } Y \in \mathcal{F}}{k}$$

TURAN GRAPH $T(n,k)$ .. NO CLIQUE OF ORDER k

$$\|T(n,k)\| = \begin{cases} \binom{n}{2} & \text{FOR } n \leq k-1 \\ \|T(n-(k-1),k)\| + (n-k+1)(k-1) + \binom{k-1}{2} \end{cases}$$

THEOREM (TURAN) : LET G BE n-VERTEX GRAPH WITH NO $K_k \Rightarrow \|G\| \leq \|T(n,k)\|$.

FIXED k, INDUCTION ON n.

G HAS K_{k-1}, INDUCTION ON $G - K_{k-1}$

K_{k-1} CLIQUE

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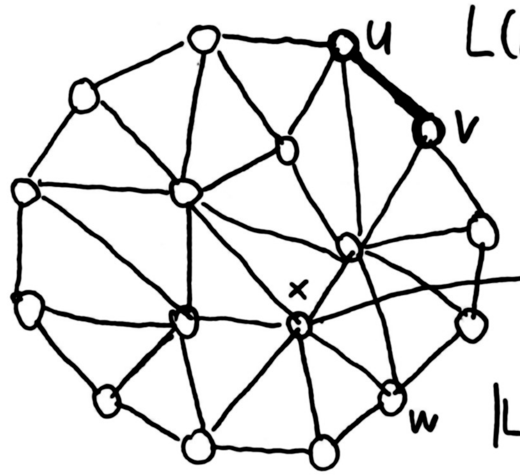
LIST COLORING $L: V_G \to \mathcal{P}(C)$
CHOOSEABILITY OF G $\quad ch(G) = \chi_\ell(G)$

A GRAPH G IS CALLED k-DEGENERATE $\Leftrightarrow$
$\forall H \subseteq G\ \exists u \in V(H): \deg_H u \leq k \quad \Leftrightarrow \qquad ch(G) \leq k+1$
$\exists$ ORDERING $v_1, \dots, v_n$: $\forall i \in [n]: |N(v_i) \cap \{v_1, \dots, v_{i-1}\}| \leq k$.

Δ-FREE PLANAR GRAPHS ARE 3-DEGENERATE
OUTERPLANAR GRAPHS ARE 2-DEGENERATE
EVERY PLANAR GRAPH IS 5-CHOOSABLE



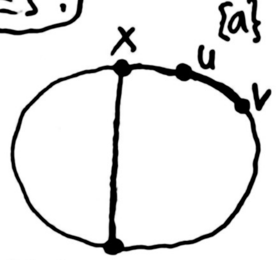
$L(u) = \{a\}$
$L(v) = \{b\}$
$|L(x)| = 5$
$|L(w)| = 3$

OUTER-TRIANGULATION $\quad G$

INDUCTION ON $|G|$

TWO CASES:

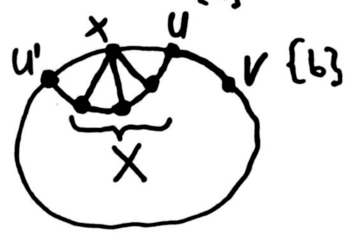
a)



DIAGONAL

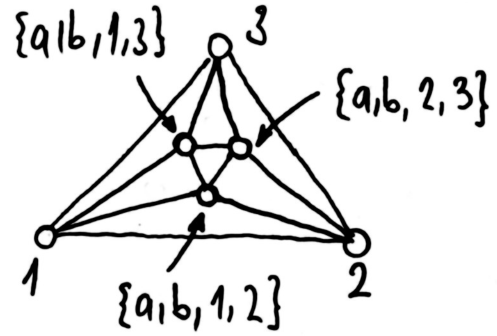
b)

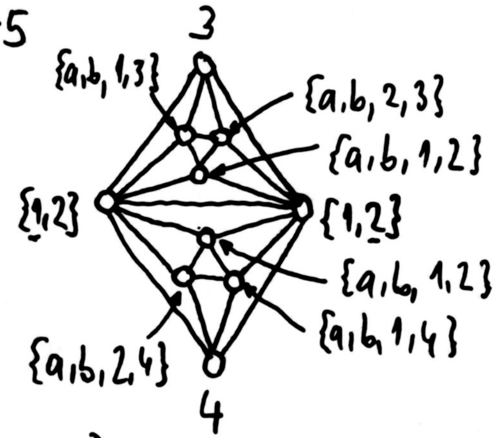
$\underbrace{c, d}_{} \neq a \to$ FORBIDDEN COLORS FOR X



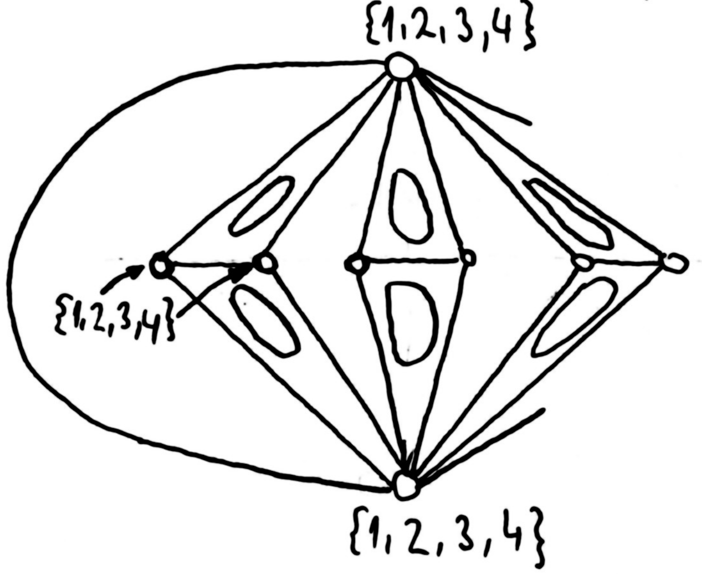
NO DIAGONAL

Ch(PLANAR GRAPHS) ≥ 5





WEDGE



... 12 WEDGES

$12 \cdot 8 + 2 = \underline{\underline{98}}$ VERTICES

THEOREM: IF G HAS AN ORIENTATION $\vec{G}$ SUCH THAT:

1) $\deg^+_{\vec{G}} u \leq k \quad \forall u \in V(G)$
2) $\forall \vec{H} \subseteq \vec{G}$: $\vec{H}$ HAS A KERNEL

$\Rightarrow \quad ch(G) \leq k+1$

$V_c \overset{def}{:=} \{x \mid c \in L(x)\}$, $\vec{G}_c \overset{def}{:=} \vec{G}[V_c]$, A_c BE A KERNEL OF $\vec{G}_c$

$x \in A_c$: $f(x) := c$

$x \notin A_c$: $L(x) := L(x) \setminus \{c\}$

NO LIST $L(u)$ BECOMES EMPTY

IT IS CLEAR FROM $|L(u)| \geq k+1$, $\deg^+_{\vec{G}} u \leq k$

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SYLABUS

KOMBINATORIKA A GRAFY I (31)

09.10.2007 TUTTE'S THEOREM +EDMOND'S ALGORITHM
[GRAPH THEORY (GT)]: 2.1.1. KÖNIG (30) → 2.1.2. HALL (31),
2.2.1. TUTTE (35), SPECIAL THM 2.2.3. (36)

16.10.2007 GALLAI-MILGRAM'S THM, DILWORTH'S THM,
NASH-WILLIAMS THM WITHOUT PROOF!
[GT]: 2.3.1. GALLAI & MILGRAM (39), 2.3.2. DILWORTH (40)
3.5.1. NASH-WILLIAMS (58)

23.10.2007 RESTRICTED SAT INSTANCES*, LOVASZ LOCAL LEMMA
[EXTREMAL COMBINATORICS [EC]: 19.1. ERDÖS-LOVÁSZ (237)

30.10.2007 MADER'S THEOREM, k-LINKED GRAPHS
[GT]: 3.3.1. MENGER (50) → 3.3.5. GLOBAL VERSION OF MENGER'S THM (55)
~~3.4.1. MADER'S THM (57)~~, 3.6.1. MADER (61) →
3.6.2 JUNG; LARMAN & MANI (62)

06.11.2007 SUNFLOWER LEMMA, ERDÖS-KO-RADO THM, TURAN'S THM
[EC]: 7.1. SUNFLOWER LEMMA (79), 8.1. ERDÖS-KO-RADO THM (89)
[EC]: 4.7. TURAN THM (42), [GT]: 7 SUBSTRUCTURES IN DENSE GRAPHS

13.11.2007 LIST COL. (CHOOSABILITY), THOMASSEN'S THM, KERNEL
[GT]: 5 COLOURING (96), LIST COLOURING (105), 5.4.2. THOM. THM (106)
KERNEL (108), LEMMA 5.4.3. (109),

20.11.2007 VIZING'S THEOREM, GALVIN'S THM,
MICIELSKI CONSTRUCTION*
[GT]: 5.3.2. VIZING (103), 5.4.4. GALVIN (109),

27.11.2007 $\boxed{\text{GRAPH MINORS}}$, $\boxed{\text{PARTIAL K-TREES}^*}$, $\boxed{\text{SERIES-PARALLEL G.}^*}$

[GT]: TREE-DECOMPOSITIONS (255), 8.3. HADWIGER'S CONJECTURE (181)

04.12.2007 $\boxed{\text{SZEMEREDI REG. LEMMA}}$, $\boxed{\text{REG. GRAPH}}$, $\boxed{\text{REMOVAL LEMMA}^*}$

[GT]: REGULARITY LEMMA (153),

11.12.2007 $\boxed{\text{ERDÖS - STONE THM}}$ [GT]: 7.1.1. TURAN THM (161)

[GT]: 7.1.2. ERDÖS & STONE (151), LEMMA 7.3.1, 7.3.2, 7.1.2 (163)

18.12.2007 $\boxed{\text{GRAPH ON SURFACES}^*}$, $\boxed{\text{HEAWOOD'S FL.}^*}$

$\boxed{\text{RAMSEY THEOREMS}^*}$

[EC]: 27. RAMSEY'S THEOREM (321)

08.01.2008 $\boxed{\text{HALES - JEWETT'S THEOREM}}$,

$\boxed{\text{VAN DER WAERDEN'S THM}}$, $\boxed{\text{GALLAI-WITT'S THM}}$

[EC]: 29. HALES - JEWETT THM (337)

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[GT]: 2. MATCHING : 2.1.1. KÖNIG, 2.1.2. HALL (2 PROOFS),
2.2.1. TUTTE, THEOREM 2.2.3, 2.3.1. GALLAI & MILGRAM,
2.3.2. DILWORTH

1. TASK - STATE THE THEOREMS

2.1.1. KÖNIG: LET $G=(V,E)$ BE A BIPARTITE GRAPH WITH PARTITIONS $\{A,B\}$. THEN THE MAXIMUM CARDINALITY OF MATCHING IS EQUAL TO THE MINIMUM CARDINALITY OF VERTEX COVER.

2.1.2. HALL: G BIPARTITE, PARTITIONS $\{A,B\}$. G HAS A MATCHING OF $A \Leftrightarrow \forall S \subseteq A : |N_G(S)| \geq |S|$.

2.2.1. TUTTE: G HAS A 1-FACTOR $\Leftrightarrow \forall S \subseteq V(G) : q(G \setminus S) \leq |S|$

THEOREM 2.2.3.: LET $G=(V,E)$ BE A GRAPH. THEN THERE EXISTS $S \subseteq V$ SUCH THAT:
(1) S IS MATCHABLE TO $G \setminus S$
(2) EVERY $C \in \mathcal{C}_{G \setminus S}$ IS FACTOR-CRITICAL.

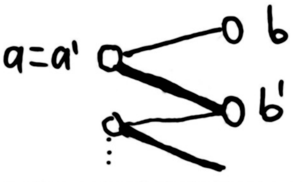
! G HAS 1-FACTOR $\Leftrightarrow |S| = |\mathcal{C}_{G \setminus S}|$

LET $S \subseteq V$ THEN H_S IS A GRAPH WITH VERTICES $S \cup \mathcal{C}_{G \setminus S}$ AND EDGES $\{ sC \mid s \in S, C \in \mathcal{C}_{G \setminus S}, \exists c \in C : sc \in E(G) \}$
S IS MATCHABLE TO $G \setminus S \Leftrightarrow \exists$ MATCHING OF S IN H_S.

PROOFS: 1.2.1. LET M BE A MATCHING WITH MAXIMUM CARD. EVERY VERTEX COVER HAS AT LEAST $|M|$ VERTICES

! [U := FOR EVERY $ab \in M$ WE CHOOSE b IFF $\exists$ ALTERNATING PATH ENDING IN b (STARTING IN UNMATCH. V. $\in A$) OTHERWISE WE CHOOSE a]

$ab \notin M$ $\exists a'b' \in M$ 1) $a' \neq a, b' = b$, ab ALTERNATING PATH $\Rightarrow b \in U$
SUPPOSE $a = a'$, $b \neq b'$, $a \notin U \Rightarrow b' \in U \Rightarrow \exists$ ALTERNATING PATH P ENDING IN b'



$\langle$ P CONTAINS $b \Rightarrow Pb$ ALT. PATH TO b
P DOES NOT $\Rightarrow Pb'a'b$, P DOES NOT CONT. $a'b'$

2.1.2.(1) $\Leftarrow$) SUPPOSE G DOES NOT HAVE MATCHING OF A

! $\Rightarrow$ MIN. VERTEX COVER $A' \cup B'$, $A' \subseteq A$, $B' \subseteq B$, $|A' \cup B'| = |A'| + |B'| < |A|$

$\Rightarrow |B'| < |A| - |A'| = |A \setminus A'|$.

THERE ARE NO EDGES $E(A \setminus A', B \setminus B') \Rightarrow N(A \setminus A') \subseteq B'$

$|N(A \setminus A')| \leq |B'| < |A \setminus A'|$. $S = A \setminus A'$.

(2) INDUCTION ON $|A|$.

IND. STEP. IF FOR EVERY $\emptyset \neq S \subset A : |N(S)| \geq |S| + 1$

WE DELETE EDGE ab AND APPLY INDUCTION

IF $\exists \emptyset \neq S \subset A : |N(S)| = |S|$

WE APPLY IND. ON $G[S \cup N(S)]$ AND $G \setminus (S \cup N(S))$

2.2.3. HOW THIS IMPLY TUTTE THM.:

IF FOR ALL $T \subseteq V$ $q(G \setminus T) \leq |T|$ (TUTTE CONDITION), THEN

FROM (1) $|S| \leq |\mathcal{C}_{G \setminus S}|$, FROM (2) $q(G \setminus S) = |\mathcal{C}_{G \setminus S}| \Rightarrow |S| = |\mathcal{C}_{G \setminus S}|$.

ALSO: IF $|S| = |\mathcal{C}_{G \setminus S}| \Rightarrow G$ HAS 1-FACTOR (APPARENTLY)

IF G HAS 1-FACTOR $\Rightarrow |S| \geq q(G \setminus S) = |\mathcal{C}_{G \setminus S}| \Rightarrow |S| = |\mathcal{C}_{G \setminus S}|$

INDUCTION ON $|G|$:

LET US FIND $S \subseteq V(G)$ SATISFYING (1) AND (2)

! LET d BE MINIMAL POSITIVE INTEGER SATISFYING FOR ANY $T \subseteq V$

(*) $$q(G \setminus T) \leq |T| + d$$

THERE EXISTS T SUCH THAT (*) HOLDS WITH EQUALITY

(IF $d > 0 \Rightarrow$ FROM MINIMALITY, IF $d = 0 \Rightarrow q(G \setminus \emptyset) \geq |\emptyset| + 0$)

LET S BE SUCH T WITH MAXIMAL CARDINALITY.

1) EVERY $C \in \mathcal{C}_{G \setminus S}$ IS ODD: IF C IS EVEN, $c \in C$, $S' := S \cup \{c\}$

$$q(G \setminus S') \geq q(G \setminus S) + 1 = |S| + d + 1 = |S'| + d \underset{(*)}{\geq} q(G \setminus S') \quad ↯$$

EVERY ODD C IN $G \setminus S$ IS ODD IN $G \setminus S'$ (*)

2) EVERY $C \in \mathcal{C}_{G \setminus S}$ IS FACTOR CRITICAL: OTHERWISE $\exists C \in \mathcal{C}_{G \setminus S}$, $c \in C$,

$C - c =: C'$ WITH NO 1-FACTOR I.E. $\exists T \subset V(C') : q(C' \setminus T) > |T|$

LET $S' := S \cup \{c\} \cup T$. EVEN $q(C' \setminus T) \geq |T| + 2$

$$q(G \setminus S') \geq q(G \setminus S) - 1 + q(C' \setminus T) \geq q(G \setminus S) + |T| + 1 \underset{(*)}{=}$$

$$= |S| + d + |T| + 1 = |S'| + d \underset{(*)}{\geq} q(G \setminus S') \quad ↯$$

3) NOW WE HAVE TO PROVE, THAT S IS MATCHABLE TO $G \setminus S$.

IF $S = \emptyset$ THEN IT IS TRIVIAL. SUPPOSE $S \neq \emptyset$, FROM (*) $\mathcal{C}_{G \setminus S}$ IS NONEMPTY

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LET $\mathcal{C} = \mathcal{C}_{G \setminus S}$. LET US CONSIDER H_S.
FOR EVERY $\mathcal{C}' \subseteq \mathcal{C}$ LET $S' = N(\mathcal{C}') \subseteq S$.

$$|S'| \underset{(*)}{\geq} \underbrace{q(G \setminus S') - d \geq |\mathcal{C}'| - d}_{\text{EVERY ODD } C \in \mathcal{C}' \text{ IS ODD IN } G \setminus S'}$$

$N(A') \geq |A'| - d$
$\Rightarrow$ MATCHING OF C. $|A| - d$

$\Rightarrow$ H_S CONTAINS MATCHING OF CARDINALITY $|\mathcal{C}| - d$
I.E. $|\mathcal{C}| - d = q(G \setminus S) - d \underset{(*)}{=} |S|$.

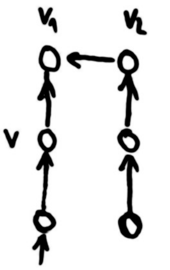
2.3.1. GALLAI & MILGRAM : FOR EVERY DIRECTED GRAPH G THERE EXISTS A PATH COVER BY AT MOST $\alpha(G)$ PATHS.
$\alpha(G)$ IS A MAXIMUM CARDINALITY OF SET OF INDEP. VERTICES

2.3.2. FOR EVERY <u>FINITE</u> PARTIALLY ORDERED SET (POSET) $(P, <)$, THE MAXIMUM LENGTH OF ANTICHAIN IS EQUAL TO THE MINIMUM NUMBER OF COVERING CHAINS. <u>(DILWORTH)</u>

PROOFS: 2.3.1. LET $\mathcal{P}_1, \mathcal{P}_2$ BE TWO ~~[illegible]~~ PATH COVERS OF G
WE WRITE $\mathcal{P}_1 < \mathcal{P}_2$ IFF $\{ter(P) : P \in \mathcal{P}_1\} \subseteq \{ter(P) : P \in \mathcal{P}_2\}$
AND $|\mathcal{P}_1| < |\mathcal{P}_2|$.

(*) IF $\mathcal{P}$ IS $<$-MINIMAL PATH COVER THEN THERE EXISTS AN INDEPENDENT SET $V = \{v_P \mid P \in \mathcal{P}\}$ OF VERTICES, WHERE $v_P \in P$ $\forall P \in \mathcal{P}$.

PROOF (INDUCTION ON $|G|$) $\mathcal{P} = \{P_1, \dots, P_m\}$ $v_i = ter(P_i)$
SUPPOSE THAT $\mathcal{P}$ IS $<$-MINIMAL PATH COVER OF G
IF $\{v_1, \dots, v_m\}$ IS INDEPENDENT THEN WE ARE DONE
OTHERWISE (WLOG) $v_2 v_1 \in E(G)$. SUPPOSE v IS PRECEED. v_1.



$\mathcal{P}' = \{P_1 v, P_2, \dots, P_m\}$ IS MINIMAL PATH-COV. ON $G \setminus v_1$.
IF $\mathcal{P}'' < \mathcal{P}'$ THEN a) SOME PATH $P \in \mathcal{P}''$ ENDS IN $v \Rightarrow \lightning$
b) SOME PATH IN $\mathcal{P}''$ ENDS IN $v_2 \Rightarrow \lightning$
c) $ter(\mathcal{P}'') \subseteq \{v_3, \dots, v_m\} \Rightarrow |\mathcal{P}''| \leq |\mathcal{P}| - 2 \Rightarrow \lightning$

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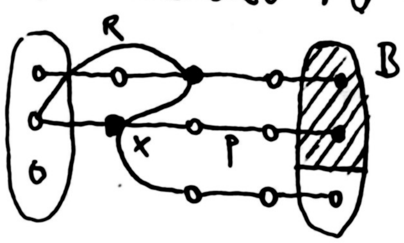
[GT]: 3.3. MENGER'S THEOREM,
3.5.1. NASH-WILLIAMS THM,
3.6. PATHS BETWEEN GIVEN PAIRS OF VERTICES

MENGER: LET $G=(V,E)$ BE A GRAPH, $A, B \subseteq V$.
THE MINIMUM CARDINALITY OF SET X SEPARATING A FROM B IS EQUAL TO THE MAXIMUM NUMBER OF DISJOINT A-B PATHS.

PROOF. LET $k = k(G,A,B)$ DENOTE THE MINIMUM NUMBER OF VERTICES SEPARATING A FROM B IN G.
WE PROVE: IF $\mathcal{P}$ IS A SET OF DISJOINT A-B PATHS, $|\mathcal{P}| < k$, THEN THERE EXISTS A SET $\mathcal{P}'$ OF DISJOINT A-B PATHS, $|\mathcal{P}'| = |\mathcal{P}|+1$ AND endvertices of $\mathcal{P}$ $\subseteq$ endvertices of $\mathcal{P}'$.
PROOF. LET $G=(V,E)$ AND A BE FIXED. INDUCTION ON $|G-B|$.
LET $\mathcal{P}$ BE A SET OF $< k$ DISJOINT A-B PATHS.
LET R BE ANY A-B PATH DISJOINT WITH VERTICES OF B THAT BELONG TO SOME PATH IN $\mathcal{P}$. ($|B| > |\mathcal{P}|$, OTHERWISE TAKE $X = B$ AS SEPARATING VERT.)



LET x BE THE LAST VERTEX ON R THAT BELONGS TO SOME PATH FROM $\mathcal{P}$.
$B' := B \cup V(xP \cup xR)$ AND APPLY INDUCTION

APPARENTLY $k(G,A,B') \geq k(G,A,B)$, BECAUSE IF X SEPARATES A FROM B', THEN ANY A-B PATH CAN BE SHORTENED TO A-B' PATH $\Rightarrow$ X SEPARATES A FROM B.

WE SAY THAT $G=(V,E)$ IS k-LINKED IFF $|G| \geq 2k$ AND FOR ALL DISTINCT VERTICES $s_1, \dots, s_k, t_1, \dots, t_k$ $\exists$ DISJOINT s_i-t_i PATHS.

3.6.1. MADER: THERE EXISTS A FUNCTION $h: \mathbb{N} \to \mathbb{N}$ THAT FOR ALL GRAPHS $G=(V,E)$, IF $d(G) \geq h(r)$ $\Rightarrow$ IN G THERE IS A TOPOLOGICAL MIRROR OF K^r AS A SUBGRAPH.

3.6.2. JUNG; LARMAN & MANI: THERE IS A FUNCTION $f: \mathbb{N} \to \mathbb{N}$ THAT EVERY $f(k)$-CONNECTED GRAPH IS k-LINKED.

3.6.1. PROOF: FOR $r=1,2$, $h(r)=1$. SUPPOSE $r>2$.

WE PROVE THAT FOR $m=r, \dots, \binom{r}{2}$ G WITH $d(G) \geq 2^m$ CONTAINS A TOPOLOGICAL MIRROR OF A GRAPH WITH r VERTICES AND m EDGES $\Rightarrow$ FOR $m=\binom{r}{2}$ A TOP. MIRROR OF K^r

INDUCTION ON m. FOR $m=r$ IF $d(G) \geq 2^m$

$\Rightarrow$ $\exists$ SUBGRAPH H OF G WITH $\delta(H) > \varepsilon(H) \geq \varepsilon(G) \geq 2^{m-1}$

$\Rightarrow$ H CONTAINS A CYCLE OF LENGTH $\delta(H)+1 > 2^{m-1}+1 \geq r+1$

THIS CYCLE IS A TOPOLOGICAL MIRROR OF C_r.

SUPPOSE THAT $m>r$, $d(G) \geq 2^m \Rightarrow \varepsilon(G) \geq 2^{m-1}$

SINCE G CONTAINS A COMPONENT C WITH $\varepsilon(C) \geq 2^{m-1}$ WE MAY SUPPOSE THAT G IS CONNECTED.

<u>LET U BE MAXIMAL CONNECTED SET OF VERTICES IN G SUCH THAT $\varepsilon(G/U) \geq 2^{m-1}$.</u> LET $H = G[N(U)]$. $N(U) \neq \emptyset$, SEC. G IS CONNECTED

IF FOR SOME $v \in H$ $d(v) < 2^{m-1}$ WE COULD ADD THIS v TO U.

$\Rightarrow$ FOR ALL $v \in H$: $d(v) \geq 2^{m-1} \Rightarrow d(H) \geq \delta(H) \geq 2^{m-1}$

$\Rightarrow$ H CONTAINS A TOPOLOGICAL MIRROR WITH r VERTICES AND $m-1$ EDGES ...

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3.6.2. PROOF: $f(k) = h(3k) + 2k$

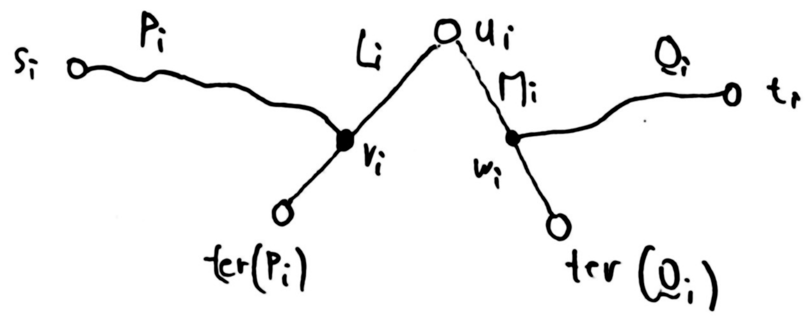
SINCE $d(G) \geq \delta(G) \geq \lambda(G) \geq \kappa(G) \geq h(3k)$

$\exists$ TOPOLOGICAL MIRROR $K \subseteq G$ OF K^{3k} WITH $3k$ BRANCH V. U

SINCE $d(G) \geq 2k$ THEN FOR DISTINCT VERTICES

$s_1, \dots, s_k$; $t_1, \dots, t_k$ $\exists$ $\mathcal{P}$ OF $2k$ DISJOINT $\{s_i, t_i\}$ - U PATHS IN G.

LET $\mathcal{P}$ BE SUCH SET THAT MINIMIZES THE NUMBER OF EDGES OUTSIDE K. AND $u_1, \dots, u_k \in U$ BE NO ENDVERTICES OF PATHS IN $\mathcal{P}$



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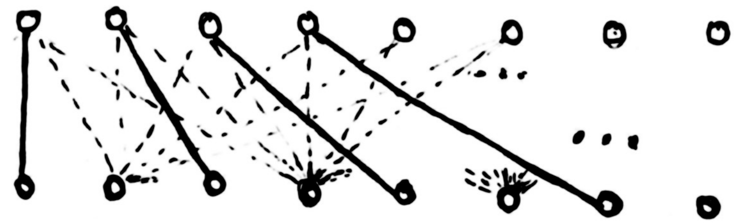
EXERCISES [GT]

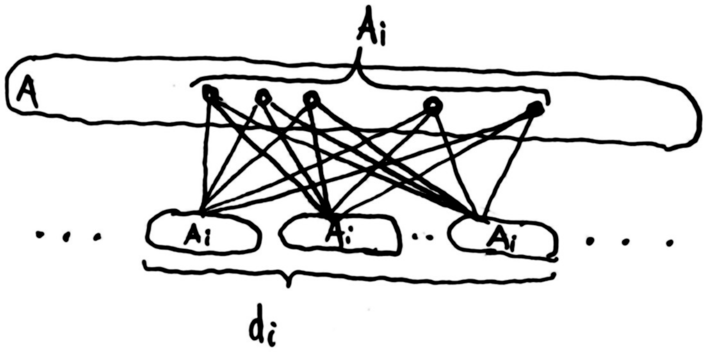
2.1. IF MATCHING M OF G CONTAINS FEWER EDGES THAN M'
$M \oplus M'$ HAS $\deg \{ \begin{smallmatrix} 0 \\ 1 \\ 2 \end{smallmatrix}$ CONSISTS OF PATHS, EVEN CYCLES AND SINGLE VERTICES. THERE EXISTS AN ODD PATH THAT IS AUGUMENTING IN G, OTHERWISE $\neg\, |M'| > |M|$.

2.3. INFINITE COUNTEREXAMPLE TO THE MARRIAGE THEOREM.



2.4.

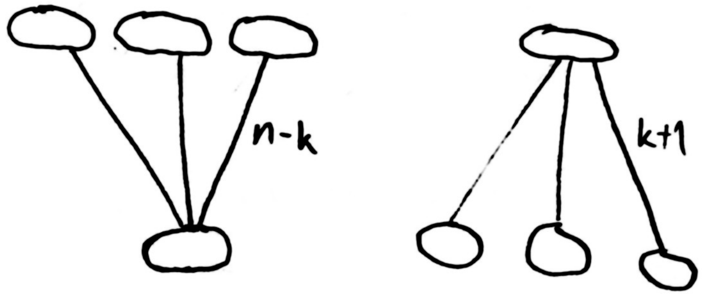
2.5. $\left|\bigcup_{i\in I} A_i\right| \geq \sum_{i\in I} d_i$



2.6.

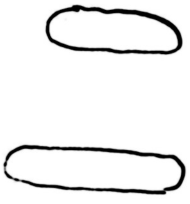
(k+1)-SUBSETS

k-SUBSETS



r

l



$r < l$

$k < \frac{n}{2}$

$n-k > \frac{n}{2} > k \Rightarrow \underline{\underline{n-k \geq k+1}}$

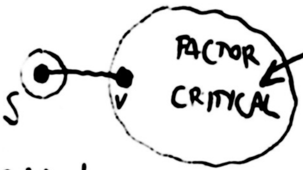
$l \cdot (n-k) = r(k+1)$

$l \cdot (n-k) \geq l(k+1) > r(k+1)$ ↯

2.7. ?

2.8. $|S| = 0 \Rightarrow$ G IS FACTOR-CRITICAL, ODD, AT LEAST $2k+1$ V.

$|S| = 1 \Rightarrow$



AT LEAST $2k-2$ VERTICES

OTHERWISE $|S| \geq k$.

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7.1. SUNFLOWER LEMMA: LET $\mathcal{F}$ BE A SET OF s-SETS. IF $|\mathcal{F}| > s!(k-1)^s$, THEN IN $\mathcal{F}$ THERE IS A SUNFLOWER WITH k PETALS.

IF $2k \leq n$ — INTERSECTING FAMILY

8.1. ERDŐS-KO-RADO: LET $\mathcal{F}$ BE A SET OF PAIRWISE NONDISJOINT k-SETS $\subseteq$ n-SET X. $\Rightarrow |\mathcal{F}| \leq \binom{n-1}{k-1}$.

4.7. TURAN THEOREM: IF $G = (V, E)$ IS WITHOUT $(k+1)$-CLIQUE $\Rightarrow |E| \leq \left(1 - \frac{1}{k}\right)\frac{n^2}{2}$

MANTEL THEOREM: IF G ON $2n$ VERTICES IS Δ-FREE $\Rightarrow |E| \leq n^2$.

7.1. PROOF: (INDUCTION ON s) FOR $s=1$, $|\mathcal{F}| > k-1$. OK ✓
LET A BE MAXIMAL SET OF DISJOINT SETS OF $\mathcal{F}$
IF $|A| \geq k$ WE ARE DONE. SUPPOSE $|A| \leq k-1$,
LET US DENOTE $B = \bigcup A$. B HAS INTERSECTION WITH EVERY SET $\mathcal{F}$. $|B| \leq s.(k-1)$
$\Rightarrow \exists x \in B$ IS CONTAINED IN AT LEAST $\frac{|\mathcal{F}|}{|B|} > (s-1)!(k-1)^{s-1}$ SETS IN $\mathcal{F}$ $\Rightarrow$ AND WE CAN APPLY INDUCTION. □

8.1. PROOF: LET US DENOTE $B_s = \{s, s+1, \ldots, s+k-1\}$ (mod n)
LET $\mathcal{F}_0$ BE A INT. FAMILY OF SUCH SETS, SUPPOSE $B_0 \in \mathcal{F}_0$
IN $\mathcal{F}_0$ CAN BE SETS B_s FOR $-(k-1) \leq s \leq k-1$ $s \neq 0$ $\equiv 2k-2$ SETS
SETS B_s, B_{s+k} ARE DISJOINT FOR $-(k-1) \leq s \leq -1$ ($k-1$ PAIRS)
$\Rightarrow |\mathcal{F}_0| \leq k$.
FOR $\mathcal{F}$ GIVEN, COUNT # PAIRS (f, s) S.T. $f(B_s) \in \mathcal{F}$, $f \in S_n$
FOR GIVEN f AT MOST $k-1$ PAIRS $\Rightarrow \# \leq n!\,k$
FOR GIVEN $F \in \mathcal{F}$ WE CHOOSE s (n WAYS), f ($k!(n-k)!$ WAYS)
$\Rightarrow \# = |\mathcal{F}|.n.k!(n-k)!$ $\quad |\mathcal{F}| \leq \frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$. □

4.7. PROOF (INDUCTION ON n): G BE EDGE-MAXIMAL WITHOUT $(k+1)$-CLIQUE

LET A BE k-CLIQUE $\quad H = G - A$

BY INDUCTION $e_H \leq \left(1-\frac{1}{k}\right)\frac{(n-k)^2}{2}$

$e_A = \binom{k}{2}$, $\quad e_{H,A} \leq (n-k)\cdot(k-1)$

$e_G \leq e_H + e_A + e_{H,A}$ $\square$

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PLAN 15.05.2008

VERTEX COLOURING, k-COLOURING, CHROMATIC NUMBER, k-CHROMATIC, k-COLOURABLE, EDGE COLOURING, COL. NUMBER CHROMATIC INDEX, LIST COLOURING, LIST-CHROMATIC NUMBER, CHOICE NUMBER, LIST-CHROMATIC INDEX

[GT]: 5.1.2. FIVE COLOUR THEOREM
5.2.2. $\chi(G)$, $col(G)$
5.2.3. $\chi(G)$, δ
5.2.4. BROOKS THEOREM (WITHOUT PROOF)
5.3.1. KÖNIG THEOREM
5.3.2. VIZING THEOREM
5.4.2. THOMASSEN THEOREM
LIST COLOURING CONJECTURE
5.4.3. KERNEL + LIST COLOURING
5.4.4. GALVIN THEOREM + COROLLARY

5.1.2. EVERY PLANAR GRAPH IS 5-COLOURABLE

5.2.2. $col(G)$ COLOURING NUMBER OF G IS THE LEAST INTEGER k SUCH THAT THERE EXISTS A VERTEX ENUMERATION $v_1, \dots, v_n$ OF G S.T. EVERY VERTEX v_i HAS LESS THAN k NEIGHBOURS IN $\{v_1, \dots, v_{i-1}\}$.

APPARENTLY $col(G) \leq \max_{H \subseteq G} \delta(H) + 1$

BUT ALSO FOR $H \subseteq G$: $col(G) \geq col(H)$

AND $col(H) \geq \delta(H)+1$ SINCE EVERY VERTEX ENUMERATION $w_1, \dots, w_m$ OF H ENDS IN w_m AND $d(w_m) \geq \delta(H)$.

$\Rightarrow$ $col(G) = \max_{H \subseteq G} \delta(H) + 1$. APPARENTLY $\underline{\underline{\chi(G) \leq col(G)}}$.

5.2.3. FOR EVERY G $\exists H \subseteq G$: $\delta(H) \geq \chi(G)-1$

5.2.4. IF CONNECTED G IS NOT COMPLETE, NOR AN ODD CYCLE $\Rightarrow \chi(G) \leq \Delta(G)$.

5.3.1. FOR EVERY BIPARTITE G : $\chi'(G) = \Delta(G)$.

5.3.2. FOR EVERY GRAPH G: $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$.

5.4.2. EVERY PLANAR GRAPH IS 5-CHOOSABLE

LIST COLOURING CONJECTURE : $ch'(G) = \chi'(G)$

IF D IS A DIRECTED GRAPH. INDEPENDENT $U \subseteq V(D)$ S.T. EVERY VERTEX IN $D-U$ SENDS AN EDGE TO U IS CALLED A KERNEL.

5.4.3. LET $G=(V,E)$ BE A GRAPH AND $(S_v)_{v\in V}$ BE VERTEX LISTS. IF THERE EXISTS AN ORIENTATION D OF G S.T. $\forall v \in D : d^+(v) < |S_v|$ AND EVERY INDUCED GRAPH OF D HAS A KERNEL, THEN G IS COLOURABLE FROM LISTS (S_v).

5.4.4. FOR EVERY BIPARTITE GRAPH $ch'(G) = \chi'(G)$.

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PLAN 17.05.2008

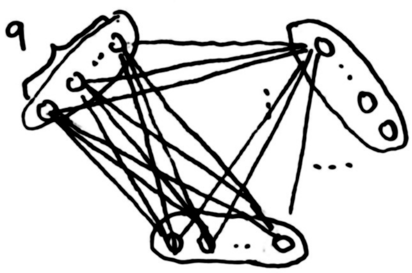
SZEMERÉDI'S REGULARITY LEMMA (MAIN GOAL)

EXERCISE:

$t_{r-1}(n) \leq \frac{1}{2} n^2 \frac{r-2}{r-1}$ WITH EQUALITY WHENEVER $r-1 \mid n$.

$n = (r-1)q + p \qquad 0 \leq p < r-1$

FOR $p = 0$:

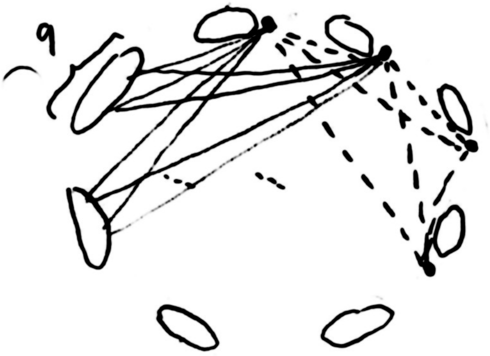


$r-1$ PARTITIONS

INDUCTION

$n = r-1 \Rightarrow \frac{1}{2} n^2 \frac{r-2}{r-1} = \frac{1}{2} n(n-1)$ ✓

$$t_{r-1}(n) = t(n-r+1) + (n-r+1)(r-2) + \binom{r-1}{2} =$$
$$= \frac{1}{2}(n-r+1)^2 \frac{r-2}{r-1} + (n-r+1)(r-2) + \frac{(r-1)(r-2)}{2} =$$
$$= \frac{1}{2}\left[(n-r+1)^2 + 2(n-r+1)(r-1) + (r-1)^2\right]\frac{r-2}{r-1} =$$
$$= \frac{1}{2} n^2 \frac{r-2}{r-1} \quad \square$$



$r-1$ PARTITIONS

$$t_{r-1}(\overbrace{(r-1)q+p}^{n}) =$$
$$= t_{r-1}((r-1)q) + p\cdot[(r-1)q - q] + \binom{p}{2} =$$
$$= \frac{1}{2}(r-1)^2 q^2 \frac{r-2}{r-1} + p\cdot(r-2)q + \frac{p(p-1)}{2} =$$
$$= \frac{1}{2}\left[(r-1)^2 q^2 + 2(r-1)q\cdot p + p^2\right]\frac{r-2}{r-1} - \frac{1}{2}\frac{r-2}{r-1}p^2 + \frac{p(p-1)}{2}$$

$$\frac{p(p-1)}{2} < \frac{1}{2}\frac{r-2}{r-1}p^2$$
$$(r-1)p^2 - (r-1)p < (r-2)p^2$$
$$p^2 < (r-1)p$$
✓

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EXTREMAL (NUM. OF EDGES), TURÁN GRAPHS
DENSITY OF THE PAIR $d(X,Y)$ $X,Y \subseteq V$ ARE DISJOINT
ε-REGULAR PAIR (A,B) OF DISJOINT SETS $A,B \subseteq V$
EXCEPTIONAL SET, ε-REGULAR PARTITION OF G,
REGULARITY GRAPH OF G WITH PARAM. ε, ℓ AND d.

TASKS:

7.1.1. TURÁN'S THEOREM
BOUND ON THE NUMBER OF EDGES OF TURÁN'S GRAPH
7.1.4. LEMMA ON DENSITY OF EDGES OF TURÁN'S GRAPH
7.2.1. REGULARITY LEMMA
7.3.1. LEMMA ON ε-REGULAR PAIR
7.3.2. LEMMA ON REGULARITY GRAPH
7.1.2. ERDŐS & STONE THEOREM
7.1.3. COROLLARY OF 7.1.2.

7.1.1. LET $n, r \in \mathbb{N}$ $r \geq 2$. EVERY GRAPH $G \not\supseteq K^r$ ON n VERTICES WITH $ex(n, K^r)$ EDGES IS $T^{r-1}(n)$.

$t_{r-1}(n) \leq \frac{1}{2} n^2 \frac{r-2}{r-1}$ WITH EQUALITY IF $r-1 \mid n$.

7.1.4. $\lim_{n \to \infty} t_{r-1}(n) \cdot \binom{n}{2}^{-1} = \frac{r-2}{r-1}$

7.2.1. FOR EVERY $m \geq 1$, $\varepsilon > 0$ $\exists M \geq m$ SUCH THAT EVERY GRAPH WITH $\geq m$ VERTICES HAS ε-REGULAR PARTITION $\{V_0, V_1, \dots, V_k\}$ WITH $m \leq k \leq M$.

7.3.1. IF (A,B) IS AN ε-REGULAR PAIR WITH $d = d(A,B)$, $Y \subseteq B$, $|Y| \geq \varepsilon |B|$ THEN ALL BUT AT MOST $\varepsilon |A|$ VERTICES IN A HAVE AT LEAST $(d-\varepsilon)|Y|$ NEIGHBOURS IN Y.

7.3.2. FOR EVERY $d \in (0;1\rangle$ AND $\Delta \geq 1$ THERE EXISTS $\varepsilon_0 > 0$ SUCH THAT FOR ANY GRAPH G AND H WITH $\Delta(H) \leq \Delta$ AND $s \in \mathbb{N}$ IF R IS REGULARITY GRAPH OF G WITH PARAMETERS $\varepsilon \leq \varepsilon_0$, $\ell \geq s/\varepsilon_0$ AND d THEN

$$H \subseteq R_s \Rightarrow H \subseteq G$$

7.1.2. FOR EVERY $r \geq 2$, $s \geq 1$ AND $\varepsilon > 0$ THERE EXISTS $n_0 \in \mathbb{N}$ SUCH THAT FOR EVERY GRAPH ON $n \geq n_0$ VERTICES AND AT LEAST $t_{r-1}(n) + \varepsilon n^2$ EDGES : $K_s^r \subseteq G$.

7.1.3. FOR EVERY GRAPH H WITH AT LEAST ONE EDGE:

$$\lim_{n \to \infty} ex(n, H) \binom{n}{2}^{-1} = \frac{\chi(H) - 2}{\chi(H) - 1}.$$

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7.3.2. PROOF $d \in (0;1\rangle$, $\Delta \geq 1$ AND $s \geq 1$ ARE GIVEN.
SUPPOSE THAT $\varepsilon_0 > 0$ SUCH THAT

$$\boxed{\frac{(\Delta+1)\,\varepsilon_0}{(d-\varepsilon_0)^{\Delta}} \leq 1}$$

LET R BE A REGULARITY GRAPH OF G WITH PARAMETERS $\varepsilon \leq \varepsilon_0$, $\ell \geq s/\varepsilon_0$ AND d, $H \subseteq R_s$, $\Delta(H) \leq \Delta$
$u_1, \dots, u_h$ ARE VERTICES OF H,
u_i BELONGS TO SOME V_j^s, WE HAVE A MAP $\sigma: i \mapsto j$
FOR EVERY $i = 1..h$ WE WANT TO FIND $v_i \in V_{\sigma(i)}$ SUCH THAT IF $u_j u_i$ IS AN EDGE OF H, THEN $v_j v_i$ IS AN EDGE OF G.

FIRST WE SET $Y_i^0 := V_{\sigma(i)}$ FOR ALL $i = 1, \dots, h$.
IN j-TH STEP WE CHOOSE $v_j \in Y_j^{j-1}$,
SET $Y_j^j := \{v_j\}$ AND FOR ALL EDGES $u_j u_i$ FOR $i > j$ WE DELETE FROM Y_i^{j-1} ALL VERTICES THAT ARE NOT NEIGHBOURS OF v_j IN G, I.E.

$$Y_i^j = N(v_j) \cap Y_i^{j-1} \quad \text{FOR ALL } i > j,\ u_j u_i \in E(H)$$

LET $A = V_{\sigma(j)}$, $B = V_{\sigma(i)}$, $Y_i^{j-1} \subseteq B$. IF $|Y_i^{j-1}| \geq \varepsilon \ell$ THEN ALL BUT AT MOST $\varepsilon |A|$ VERTICES IN A HAVE AT LEAST $(d-\varepsilon)|Y_i^{j-1}|$ NEIGHBOURS IN Y_i^{j-1}
THERE ARE AT MOST Δ INDICES $i > j$ SUCH THAT $u_j u_j \in E(H)$
SO EVERY Y_i IS SHRINKED AT MOST Δ TIMES
WE REQUIRE $|Y_i^{j-1}| - \Delta \varepsilon \ell \geq (d-\varepsilon)^{\Delta} \ell - \Delta \varepsilon \ell \geq$
$\geq (d-\varepsilon_0)^{\Delta} \ell - \Delta \varepsilon_0 \ell \geq \varepsilon_0 \ell \geq s$ I.E. $\frac{(\Delta+1)\varepsilon_0}{(d-\varepsilon_0)^{\Delta}} \leq 1$

SO IN PARTICULAR $|Y_j^{j-1}| - \Delta \varepsilon \ell \geq s$ AND $|Y_i^{j-1}| \geq \varepsilon_0 \ell \geq \varepsilon \ell$

7.1.2 PROOF WE ARE GIVEN $\gamma > 0$ AND WANT TO PROVE THAT FOR ALL SUFFICIENTLY LARGE n ALL GRAPHS WITH n VERTICES AND AT LEAST $t_{r-1}(n) + \gamma n^2$ EDGES CONTAIN K_s^r AS A SUBGRAPH WHERE $r \geq 2$, $s \geq 1$.

FOR $s=1$ THIS FOLLOWS FROM TURÁN'S THEOREM
SUPPOSE THAT $s \geq 2$.

OUR PLAN IS TO USE REGULARITY LEMMA SO THAT ANY GRAPH G WITH SUFFICIENTLY LARGE ORDER HAS A REGULARITY GRAPH R
PROVIDED WITH LARGE $\|G\|$ WE HOPE THAT R WILL HAVE ENOUGH EDGES TO CONTAIN K^r AS A SUBGRAPH (FROM TURÁN'S THEOREM) SO $K^r \subseteq R \Rightarrow K_s^r \subseteq R_s \Rightarrow K_s^r \subseteq G$

FOR $d := \gamma$ AND $\Delta := \Delta(K_s^r)$ WE GET FROM 7.1.2. $\varepsilon_0 > 0$
SUPPOSE THAT $\varepsilon_0 < \frac{\gamma}{2} < 1$. LET $m > \frac{1}{\gamma}$

LET $\varepsilon > 0$ BE SUCH THAT $\varepsilon \leq \varepsilon_0$ AND

$$\boxed{2\gamma - \varepsilon^2 - 4\varepsilon - d - \frac{1}{m} > 0}$$

FOR m AND $\varepsilon > 0$ WE GET M FROM REGULARITY LEMMA.

SUPPOSE THAT $n \geq \boxed{\dfrac{M \cdot s}{\varepsilon_0(1-\varepsilon)}}$

G BE ANY GRAPH ON n VERTICES WITH AT LEAST $t_{r-1}(n) + \gamma n^2$ EDGES

SINCE $n \geq m$ THERE IS A ε-REGULAR PARTITION OF G INTO $\{V_0, \dots, V_k\}$ $m \leq k \leq M$, $|V_i| = \ell$
$n \geq k\ell$

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$$l = \frac{n - |V_0|}{k} \geq \frac{n - \varepsilon n}{M} = n\frac{(1-\varepsilon)}{M} \geq \frac{s}{\varepsilon_0}$$

LET R BE REGULARITY GRAPH OF G WITH PARAMETERS $\varepsilon \leq \varepsilon_0$, $l \geq \frac{s}{\varepsilon_0}$ AND d CORRESPONDING TO OUR PARTITION $\{V_0, \dots, V_k\}$.

- OUR AIM IS TO PROVE THAT G HAS ENOUGH ε-REGULAR (V_i, V_j) PAIRS OF DENSITY AT LEAST d SO THAT $K^r \subseteq R$. I.E. R HAS MORE THAN $t_{r-1}(k)$ EDGES.

WE COUNT HOW MANY EDGES OF G DO NOT CONTRIBUTE TO OUR ε-REG. PAIRS.

INSIDE V_0 THERE ARE AT MOST $\binom{|V_0|}{2} \leq \frac{|V_0|^2}{2} \leq \boxed{\frac{1}{2}(\varepsilon n)^2}$ EDGES

V_0 CAN BE CONNECTED TO OTHER VERTICES WITH AT MOST $\boxed{\varepsilon n \cdot kl}$ EDGES

AT MOST εk^2 PAIRS ARE NOT ε-REGULAR $\Rightarrow$ AT MOST $\boxed{\varepsilon k^2 l^2}$ EDGES

- THERE ARE AT MOST $\boxed{\frac{1}{2} d k^2 l^2}$ EDGES OF ε-REG. PAIRS WITH DENSITY AT MOST d

INSIDE EACH V_i THERE ARE AT MOST $\frac{1}{2} l^2$ EDGES SO TOTALY AT MOST $\boxed{\frac{1}{2} k l^2}$ EDGES.

THE REST ARE ε-REG. EDGES WITH DENSITY $\geq d$

$$\Rightarrow \|G\| \leq \frac{1}{2}\varepsilon^2 n^2 + \varepsilon n k l + \varepsilon k^2 l^2 + \frac{1}{2} d k^2 l^2 + \frac{1}{2} k l^2 + \|R\| l^2$$

$$\Rightarrow \|R\| \geq k^2 \frac{\|G\| - \frac{1}{2}\varepsilon^2 n^2 - \varepsilon n k l - \varepsilon k^2 l^2 - \frac{1}{2} d k^2 l^2 - \frac{1}{2} k l^2}{k^2 l^2} \geq$$

$$\geq \frac{1}{2} k^2 \left(\frac{t_{r-1}(n) + \gamma n^2 - \frac{1}{2}\varepsilon^2 n^2 - \varepsilon n^2}{\frac{1}{2} n^2} - 2\varepsilon - d - \frac{1}{k} \right) \geq$$

$$\geq \frac{1}{2} k^2 \left(t_{r-1}(n) \binom{n}{2}^{-1} \left(1 - \frac{1}{n}\right) + \boxed{2\gamma - \varepsilon^2 - 4\varepsilon - d - \frac{1}{m}} \right) >$$

$$> \frac{1}{2} k^2 \frac{r-2}{r-1} \geq t_{r-1}(k).$$ FOR SUFFICIENTLY LARGE n. □

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NDMI012 KOMBINATORIKA A GRAFY II - SKÚŠKA

PLAN 20.05.2008

[EC]: 29.1. HALES-JEWETT THEOREM
29.3. VAN DER WAERDEN THEOREM
29.4. GALLAI-WITT'S THEOREM

29.1. HALES-JEWETT: LET A BE A FINITE ALPHABET WITH t SYMBOLS, r BE A POSITIVE INTEGER. THEN THERE EXISTS A POSITIVE INTEGER $n = HW(r,t)$ SUCH THAT FOR EVERY COLOURING $A^n \to \{1,\dots,r\}$ THERE EXISTS MONOCHROMATIC COMBINATORIAL LINE.

29.3. VAN DER WAERDEN THEOREM: FOR EVERY POSITIVE INTEGERS r AND t THERE EXISTS A POSITIVE INTEGER $N = W(r,t)$ SUCH THAT FOR EVERY r-COLOURING OF $\{1,\dots,N\}$ THERE EXISTS A MONOCHROMATIC ARITHMETIC PROGRESSION WITH t TERMS.

29.4. GALLAI-WITT'S THEOREM: LET VECTORS OF $\mathbb{Z}^m$ BE FINITELY COLORED. THEN EVERY FINITE SUBSET OF $\mathbb{Z}^m$ HAS MONOCHROMATIC HOMOTHETIC COPY.

29.3. PROOF : WE ARE GIVEN $r, t > 0$.

LET $n := HJ(r, t)$, $N := n(t-1)$.

$A := \{0, 1, \dots, t-1\}$ IS ALPHABET

FOR EVERY $x = (x_1, \dots, x_n) \in A^n$ DEFINE MAPPING

$f(x) = x_1 + \dots + x_n \in \{0, \dots, N\}$

THIS MAPPING INDUCES A COLOURING OF A^n

COLOR OF $x \in A^n$ IS THE COLOR OF $f(x)$.

THERE IS A ROOT τ SUCH THAT

$L_\tau = \{\tau(0), \dots, \tau(t-1)\}$ IS MONOCHROMATIC

$f(\tau(0)), \dots, f(\tau(t-1))$ IS MONOCHROMATIC ARITHMETIC PROGRESSION OF LENGTH t.

29.4. PROOF. FIRST FIX THE NUMBER OF COLOURS r

AND $V = \{v_0, \dots, v_{t-1}\}$.

LET $n := HJ(r, t)$, $A = V$ IS THE ALPHABET

WE DEFINE COLOURING $A^n \to \{1, \dots, r\}$

FOR $x = (x_1, \dots, x_n)$ WE DEFINE

COLOUR OF $x_1 + \dots + x_n \in \mathbb{Z}^m$.

THERE IS A ROOT τ S. T. L_τ IS MONOCHROMATIC

LET $I = \{i \in \{1, \dots, n\} \mid \tau_i = *\}$

$u = \sum_{i \in [n] \setminus I} \tau_i$, $\lambda = |I|$

APPARENTLY $u + \lambda V$ IS MONOCHROMATIC.

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29.1. HALES - JEWETT THEOREM PROOF

INDUCTION ON t. FOR $t=1$ TRIVIAL.

LET $n := HJ(r, t-1)$.

$N_1 := r^{t^n}, \dots, N_i = r^{t^{n+\sum_{j=1}^{i-1} N_j}}, \dots, N_n,$

$N := N_1 + \dots + N_n$.

WE WANT TO PROVE, THAT $HJ(r,t) \leq N$.

LET $A = \{0, \dots, t-1\}$ AND $\chi : A^N \to \{1, \dots, r\}$ BE ANY COLOURING.

WE SAY THAT $a, b \in A^n$ ARE NEIGHBOURS IF FOR SOME i

$a = a_1 \dots a_{i-1}\, 0\, a_{i+1} \dots a_n$,

$b = a_1 \dots a_{i-1}\, 1\, a_{i+1} \dots a_n$

LET $\tau = \tau_1 \dots \tau_n$ $\tau_i \in (A \cup \{*\})^{N_i}$ IS ROOT $\forall i$

DEFINE $\tau(a) = \tau_1(a_1) \dots \tau_n(a_n)$.

CLAIM: THERE ARE SUCH ROOTS $\tau_1, \dots, \tau_n$ THAT $\chi(\tau(a)) = \chi(\tau(b))$ FOR ALL NEIGHBOURS a, b

DEFINE COLOURING $\chi' : (A \setminus \{0\})^n \to \{1, \dots, r\}$ AS $\chi'(a) = \chi(\tau(a))$.

SINCE $|A \setminus \{0\}| = t-1$ AND $n = HJ(r, t-1)$, THERE EXISTS A COMBINATORIAL LINE FOR χ'

$\nu = \nu_1 \dots \nu_n \in (A \setminus \{0\} \cup \{*\})^n$

$\tau(v) = \tau_1(v_1) \ldots \tau(v_n)$ IS A ROOT $\in (A \cup \{*\})^N$

LET US PROVE THAT

$L_{\tau(v)} = \{ \tau(v(0)), \ldots, \tau(v(t-1)) \}$

IS A COMBINATORIAL LINE FOR χ

SINCE $\chi'(v(1)) = \ldots = \chi'(v(t-1))$

I.E. $\chi(\tau(v(1))) = \ldots = \chi(\tau(v(t-1)))$

IT SUFFICES TO SHOW THAT $\tau(v(0))$ HAS THE SAME COLOUR.

IF v HAS ONE WILDCARD $*$, THEN
$v(0)$ AND $v(1)$ ARE NEIGHBOURS

OTHERWISE THERE IS A SEQUENCE OF NEIGHBOURS

$$\begin{array}{ll} v(0) & \ldots 0 \ldots 0 \ldots 0 \ldots \\ & \ldots 0 \ldots 0 \ldots 1 \ldots \\ & \ldots 0 \ldots 1 \ldots 1 \ldots \\ v(1) & \ldots 1 \ldots 1 \ldots 1 \ldots \end{array}$$

SO $\chi(v(0)) = \chi(v(1))$.

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CLAIM: THERE ARE ROOTS $\tau_1, \dots, \tau_n$, $\tau_i \in (A \cup \{*\})^{N_i}$ SUCH THAT $\chi(\tau(a)) = \chi(\tau(b))$ FOR ALL NEIGHBOURS $a, b \in A^n$.

SUPPOSE THAT $\tau_{i+1}, \dots, \tau_n$ ARE DEFINED

$L_i := \sum_{j=1}^{i} N_j$

DEFINE $W_k = \underbrace{00\dots0}_{k}\underbrace{11\dots1}_{N_i - k}$

LET $\chi_0, \dots, \chi_{N_i}$ BE COLOURINGS OF $A^{L_{i-1}+n-i}$

$$\chi_k(x_1 \dots x_{L_{i-1}} y_{i+1} \dots y_n) := \chi(x_1 \dots x_{L_{i-1}} W_k \tau_{i+1}(y_{i+1}) \dots \tau_n(y_n))$$

NUMBER OF COLOURINGS $= r^{\# \text{WORDS}} = r^{t^{L_{i-1}+n-i}} \leq r^{t^{n+L_{i-1}}} = N_i$

$\Rightarrow \exists\, k < s : \chi_k = \chi_s$.

DEFINE $\tau_i = \underbrace{0\dots0}_{k}\underbrace{**\dots*}_{s-k}\underbrace{11\dots1}_{N_i - s}$

SUPPOSE

$a = a_1 \dots a_{i-1}\, 0\, a_{i+1} \dots a_n$

$b = b_1 \dots b_{i-1}\, 1\, b_{i+1} \dots b_n$

$$\chi(\tau(a)) = \chi(\tau(a_1) \dots \tau(a_{i-1})\, W_s\, \tau(a_{i+1}) \dots \tau(a_n)) =$$

$$= \chi(\quad -\parallel- \quad W_k \quad -\parallel- \quad) =$$

$$= \chi(\tau(b)). \quad \square$$

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SURFACE OF GENUS γ, $\gamma(G)$, CELLULAR EMBEDING,
HYPERGRAPH H, $\chi(H)$, $H(\gamma)$

PLAN 21.05.2008

1. LEMMA ON CELLULAR EMBEDING
2. EULER FORMULA FOR SURFACES + COROLLARY
3. HEAWOOD THEOREM
4. COMPACTNESS THEOREM
5. FINITE, INFINITE RAMSEY THEOREM

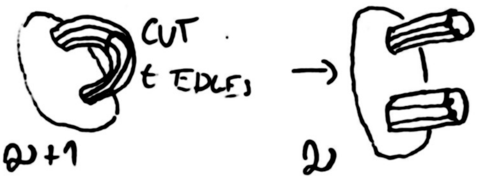
---

1. FOR EVERY G ON S_γ $\exists G' \supseteq G$ S.T.
   G' HAS CELLULAR EMBEDING AND G IS SPANNING SUBGRAPH OF G'.
2. FOR EVERY G ON S_γ, G HAS CELLULAR EMBEDING
   IF $v = |G|$, $e = \|G\|$, f IS THE NUMBER OF FACES
   $$v + f = e + 2(1-\gamma)$$
   FOR EVERY G ON S_γ : $e \leq 3v + 6(\gamma - 1)$
3. $$H(\gamma) = \max_{\gamma(G)=\gamma} \chi(G) = \left\lfloor \frac{1}{2}\left(7 + \sqrt{48\gamma+1}\right)\right\rfloor$$
4. IF H IS INFINITE HYPERGRAPH WITH FINITE EDGES, k IS POSITIVE INTEGER
   IF FOR EVERY FINITE $H' \subseteq H$: $\chi(H') \leq k$
   THEN $\chi(H) \leq k$
5. $\forall k, n, p$: $N \leftarrow (n)_p^k$, $\quad \omega \leftarrow (\omega)_p^k$.

2. $v+f=e+2(1-g)$

PROOF (INDUCTION ON g). FOR $g=0$ PLANAR GRAPHS

FOR $g+1$



$v'=v+2t$

$e'=e+t+2t$

$f'=f+t+2$

BY INDUCTION $v'+f'=e'+2(1-g) \Rightarrow v+f=e+2[1-(g+1)]$.

COUNT #(FACE, EDGE) $\# \geq 3f$, $\# = 2e$

$2e \geq 3f = 3e-3v+6(1-g)$

$\Rightarrow e \leq 3v+6(g-1)$ HOLDS FOR ALL G ON S_g

3. HEAWOOD THEOREM: $H(g)=\max\limits_{g(G)=g}\chi(G)=\left\lfloor \frac{1}{2}\left(7+\sqrt{48g+1}\right)\right\rfloor$

PROOF OF $\leq$: LET G BE ON S_g, $v=|G|$.

$\delta(G) \leq v-1$, $\sum \deg v = 2e \leq 6v+12(g-1)$

$\Rightarrow \delta(G) \leq d(G) \leq 6+12\frac{g-1}{v}$

$\delta(G) \leq \min\left\{v-1,\ 6+12\frac{g-1}{v}\right\}$ $v-1=6+12\frac{g-1}{v}$

$v^2-7v+12(1-g)=0$

$v_{1,2}=\frac{7\pm\sqrt{49-48(1-g)}}{2}=\frac{1}{2}\left(7\pm\sqrt{48g+1}\right)>0$

$\Rightarrow \delta(G) \leq \frac{1}{2}\left(7+\sqrt{48g+1}\right)-1$

FOR SOME $w \in G$: $d(w)=\delta(G)$ AND WE APPLY INDUCTION ON $G-w$.

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4. LET $H = (V, E)$, $V = \{v_1, v_2, \dots\}$, $E \subseteq P(V)$
BE INFINITE HYPERGRAPH WITH FINITE EDGES.
SUPPOSE THAT FOR EVERY FINITE $H' \subseteq H : \chi(H') \leq k$.
LET $V_i = \{v_1, \dots, v_i\}$, DEFINE $\chi_i : V_i \to \{1, \dots, k\}$.
COLOURING OF $H[V_i]$.

THERE IS A COLOUR c_1 S.T. $\chi_i(v_1) = c_1$
FOR INFINITELY MANY i ... $I_1 \subseteq \mathbb{N}$.
DEFINE $\chi(v_1) := c_1$.
ANALOGOUSLY THERE IS A COLOUR c_2 S.T.
$\chi_i(v_2) = c_2$ FOR INF. MANY $i \in I_1$ ... $I_2 \subseteq I_1$
DEFINE $\chi(v_2) := c_2$.
...
ETC.
SUPPOSE $e \in E$, $j := \max\{i : v_i \in e\}$.
FOR INF. MANY COLOURINGS χ_i $i \in I_j$
$\chi(v_1) = \chi_i(v_1), \dots, \chi(v_j) = \chi_i(v_j)$
$\Rightarrow |\chi(e)| \geq 2$.

5. $\forall k, n, p \ \exists N : \forall X, |X| = N$
$\forall \binom{X}{p} = A_1 \dot\cup \dots \dot\cup A_k \quad \exists A \in \binom{X}{n} : \binom{A}{p} \subseteq A_i$
FOR SOME $i \in \{1, \dots, k\}$ $\equiv N \to (n)_p^k$

$\forall k, p \quad \forall \binom{\omega}{p} = A_1 \dot\cup \dots \dot\cup A_k \quad \exists$ INFINITE $A \subseteq \omega$
S.T. $\binom{A}{p} \subseteq A_i$ FOR SOME $i \in \{1, \dots, k\}$
$\omega \to (\omega)_p^k$.

$\omega \to (\omega)^k_p$ PROOF:

INDUCTION ON p. FOR $p=1$ TRIVIAL.

SUPPOSE $p > 1$. $\chi : \binom{\omega}{p} \to \{1,\dots,k\}$ BE A COLOURING.

SET $x_1 := 1$ AND DEFINE $\chi' : \binom{\omega \setminus \{1\}}{p-1} \to \{1,\dots,k\}$

AS $\chi'(\{y_1,\dots,y_{p-1}\}) := \{x_1, y_1, \dots, y_{p-1}\}$.

BY INDUCTION THERE IS INFINITE $Y^{(1)} \subseteq \omega \setminus \{1\}$

S.T. FOR ALL $\{y_1,\dots,y_{p-1}\} \in Y^{(1)}$: $\chi'(\{y_1,\dots,y_{p-1}\}) = c_1$.

LET x_2 BE FIRST NUMBER IN $Y^{(1)}$. AND AGAIN

WE FIND INFINITE $Y^{(2)} \subseteq Y^{(1)} \setminus \{x_2\}$ S.T.

$\chi(\{x_2, y_1, \dots, y_{p-1}\}) = c_2$ FOR ALL $\{y_1,\dots,y_{p-1}\} \in Y^{(2)}$.

... ETC.

WE HAVE $x_1, x_2, \dots$ WITH CORRESPONDING COLOURS $c_1, c_2, \dots$

SOME COLOUR c OCCURS INFINITELY MANY TIMES

$\Rightarrow$ $A = \{x_i \mid c_i = c\}$.

INFINITE RAMSEY $\Rightarrow$ FINITE RAMSEY:

FOR EVERY N, n, p LET $H(N,n,p) =$

$= \left(\binom{[N]}{p}, \left\{ \binom{A}{p} \mid A \in \binom{[N]}{n} \right\} \right)$

APPARENTLY $N \not\to (n)^k_p \iff \chi(H(N,n,p)) > k$.

$\omega \to (\omega)^k_p \Rightarrow \forall n\ \omega \to (n)^k_p \iff \forall n\ \chi(H(\omega,n,p)) > k$

$\forall n$ $\chi(H(\omega,n,p)) > k \Rightarrow \exists$ FINITE $H' \subseteq H(\omega,n,p)$

SUCH THAT $\chi(H') > k$ (FROM COMPACTNESS)

$\Rightarrow \exists N$: $H(N,n,p) \supseteq H' \Rightarrow \chi(H(N,n,p)) > k$

$\Rightarrow N \to (n)^k_p$.

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GRAPH MINOR RELATION, MINOR, TOPOLOGICAL MINOR

PLAN FOR 22.05.2008

1.7.3. THE MINOR RELATION, TOPOLOGICAL-MINOR RELATION ARE PARTIAL ORDERINGS

1.7.2. (i) EVERY TX IS ALSO AN MX
$\Rightarrow$ EVERY TOP.-MINOR OF A GRAPH IS ALSO ITS MINOR
(ii) IF $\Delta(X) \leq 3$ THEN EVERY MX CONTAINS TX.

4.4.2. A GRAPH CONTAINS K^5 OR $K_{3,3}$ AS A MINOR
$\Leftrightarrow$ IT CONTAINS K^5 OR $K_{3,3}$ AS A TOPOLOGICAL MINOR.

HADWIGER CONJECTURE

DEFINE k-TREE, PARTIAL k-TREE, TREEWIDTH $tw(G)$, TREE DECOMPOSITION, WIDTH OF TREE DECOMPOSITION

LEMMA: A 2-TREE CAN BE GROWN FROM ANY OF ITS EDGES

CHARACTERIZE 2-TREES

PROVE $\neg K_4 \preceq G \Rightarrow \neg \chi(G) \geq 4$

KURATOWSKI THEOREM FOR OUTERPLANAR GRAPHS
([GT]: EXERCISE 4.20 (91))

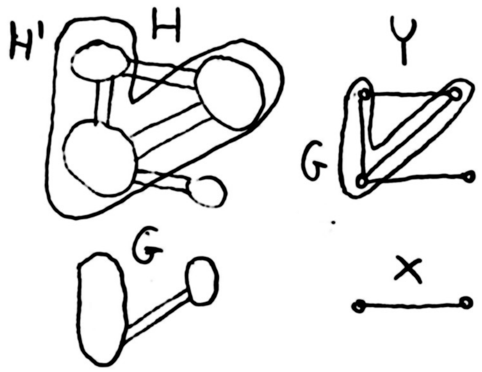
SERIES-PARALLEL GRAPHS ([GT]: EX. 8.18 (189))

FOR A GRAPH X, G IS AN MX IF THERE IS A PARTITION $\{V_x \mid x \in V(X)\}$ OF $V(G)$ SUCH THAT V_x IS CONNECTED IN G AND $xy \in E(X) \Leftrightarrow$ THERE IS AN $V_x - V_y$ EDGE IN G.
V_x ARE CALLED BRANCH SETS. INTUITIVELY X CAN BE OBTAINED FROM G BY CONTRACTING BRANCH SETS.

G IS A TX IF G CAN BE OBTAINED FROM X BY SUBDIVISION OF EDGES OF X, I.E. BY REPLACING EDGES OF X BY DISJOINT PATHS.
VERTICES OF X IN G ARE CALLED BRANCH VERTICES.

X IS A MINOR ($X \preccurlyeq Y$, $X \preccurlyeq_m Y$) OF Y $\Leftrightarrow$ $\exists G \subseteq Y$: G IS AN MX.
X IS A TOPOLOGICAL MINOR ($X \preccurlyeq_t Y$) OF Y $\Leftrightarrow$
$\exists G \subseteq Y$: G IS A TX, I.E. Y CONTAINS SUBDIVISION OF X.

$\preccurlyeq$ IS A PARTIAL ORDERING:
1) $X \preccurlyeq X$: $X \subseteq X$ AND X IS AN MX FOR $V_x = \{x\}$
2) $X \preccurlyeq Y$ & $Y \preccurlyeq X \Rightarrow X \cong Y$.
$X \preccurlyeq Y \Rightarrow |X| \leq |Y|$ & $\|X\| \leq \|Y\|$.
$\Rightarrow |X| = |Y|$ & $\|X\| = \|Y\|$.
$\Rightarrow X \preccurlyeq Y \Rightarrow$ FOR $G = Y$: G IS AN MX
SINCE $|G| = |X|$ ALL $V_x = \{v_x\}$
AND $xy \in EX \Leftrightarrow v_x v_y \in E(G)$ I.E. $G \cong X \Rightarrow X \cong Y$.
3) $X \preccurlyeq Y$ & $Y \preccurlyeq Z \Rightarrow X \preccurlyeq Z$.
$X \preccurlyeq Y \Rightarrow \exists G \subseteq Y$: G IS AN MX
$Y \preccurlyeq Z \Rightarrow \exists H \subseteq Z$: H IS AN MY



APPARENTLY WE CAN TAKE $H' \subseteq H$ S.T. H' IS AN MG. IT IS EASY TO SEE THAT H' IS MG & G IS $MX \Rightarrow H'$ IS MX. $\square$

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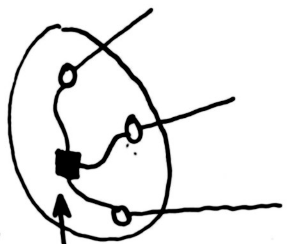
ANY MINOR X OF Y CAN BE OBTAINED FROM Y BY DELETING VERTICES, DELETING AND CONTRACTING EDGES.

IF $X \preceq Y \Rightarrow \exists G \subseteq Y : G$ IS MX. G CAN BE OBTAINED FROM Y BY DELETING VERTICES AND EDGES, X CAN BE OBTAINED FROM G BY EDGE CONTRACTION

ON THE OTHER HAND: IF WE DELETE ONE VERTEX, EDGE OR IF WE CONTRACT AN EDGE OF Y WE GET $Y' \preceq Y$. THE REST FOLLOWS FROM THAT $\preceq$ IS PARTIAL ORDERING.

1.7.2. (ii) IF $\Delta(X) \leq 3 \Rightarrow$ EVERY MX CONTAINS A TX.
SUPPOSE THAT $\Delta(X) \leq 3$ AND G IS AN MX
EVERY V_x IS CONNECTED IN G



V_x

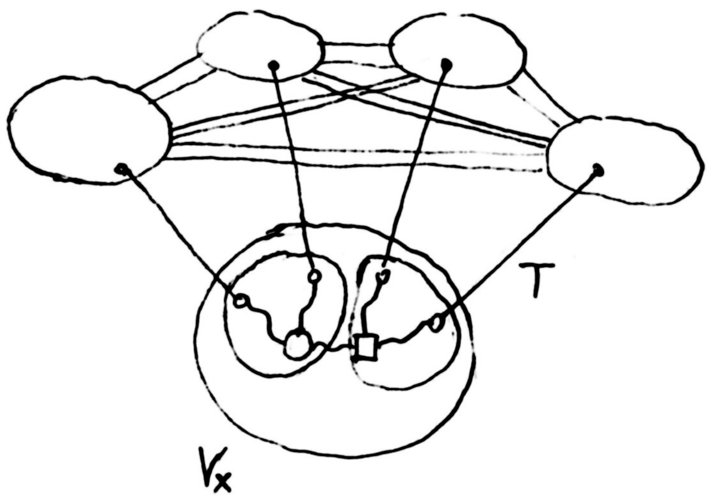
THERE IS ALWAYS A SQUARE VERTEX THAT REPRESENT BRANCH VERTEX FOR TX.

IF $\Delta(X) \leq 3 \Rightarrow X \preceq Y \Rightarrow X \preceq_t Y$.
$X \preceq Y \Rightarrow \exists G \subseteq Y : G$ IS AN $MX \Rightarrow$
G CONTAINS A TX : $\exists H \subseteq G$: H IS A $TX \Rightarrow X \preceq_t Y$.

ON THE OTHER HAND $X \preceq_t Y \Rightarrow X \preceq Y$.
$X \preceq_t Y \Rightarrow \exists G \subseteq Y : G$ IS A $TX \Rightarrow G$ IS AN $MX \Rightarrow X \preceq Y$.

4.4.2. IF G CONTAINS K^5 OR $K_{3,3}$ AS A MINOR
$\Rightarrow$ G CONTAINS K^5 OR $K_{3,3}$ AS A TOPOLOGICAL MINOR.

SINCE $\Delta(K_{3,3}) = 3$ WE MAY ASSUME THAT G CONTAINS K^5 AS A MINOR. LET $K \subseteq G$ BE SMALLEST S.T. K IS MK^5.



LET V_x BE ANY BRANCH SET

$G[V_x]$ + CROSS-EDGES IS A TREE T WITH 4 LEAVES.

IF FOR EVERY V_x THERE IS $w \in V_x$ WITH DEGREE 4, WE HAVE FOUND TK_5.
OTHERWISE THERE ARE TWO VERTICES WITH DEGREE 3.
$\Rightarrow$ APPARENTLY $K_{3,3} \preccurlyeq K \Rightarrow K_{3,3} \preccurlyeq_t K$.

HADWIGER CONJECTURE: $\chi(G) \geq k \Rightarrow K^k \preccurlyeq G$.

k-TREE:
(i) K^{k+1} IS A k-TREE
(ii) IF G IS A k-TREE. $K \subseteq G$ IS A k-CLIQUE THEN IF WE ADD w TO G AND JOIN IT TO K WE GET A k-TREE.

G IS A k-TREE $\Leftrightarrow$ $|G| \geq k+1$ AND $\exists$ ENUMERATION $v_1, \dots, v_n$ OF VERTICES OF G S.T. $G[v_1, \dots, v_{k+1}] = K^{k+1}$ AND FOR ALL $i > k+1$ $G[N_{G[v_1, \dots, v_i]}(v_i)] = K^k$.

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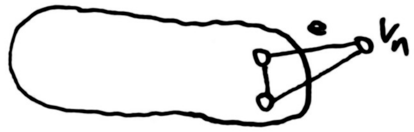
FOR $k=2$ WE CALL SUCH ENUMERATION $v_1 v_2 \cdots v_n$ A GROWING OF A 2-TREE FROM $v_1 v_2$.

LEMMA: A 2-TREE CAN BE GROWN FROM ANY OF ITS EDGES.

PROOF (INDUCTION ON $|G|$). FOR $n=3$ TRIVIAL. $n>3$ TWO CASES:

1)


2)


THEOREM: 2-TREES ARE EXACTLY EDGE-MAXIMAL GRAPHS WITHOUT K_4 MINOR.

SINCE $\Delta(K_4)=3$ $\quad K_4 \preccurlyeq G \iff K_4 \preccurlyeq_t G$

SO IT SUFFICES TO PROVE THAT 2-TREES ARE EDGE-MAXIMAL GRAPHS WITHOUT K_4 TOPOLOGICAL-MINOR.

LEMMA: EVERY 2-TREE CONTAINS AT LEAST TWO NONADJACENT VERTICES OF DEGREE 2 CONN. BY 2 DISJ. PATHS

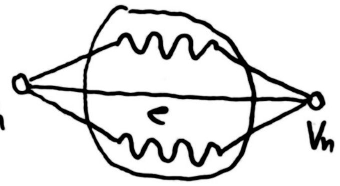
LEMMA: EVERY 2-TREE IS EDGE-MAXIMAL WITHOUT K_4 TOPOLOGICAL MINOR.

LEMMA ⇒ SUPPOSE v_1, v_n ARE NONADJACENT WITH DEGREE 2.

BY INDUCTION $G-v_n$ DOES NOT CONTAIN K_4 TOP. MINOR.

IF G CONTAINS K_4 TOP. MINOR ⇒ v_n CANNOT BE USED AS A BRANCH VERTEX, BUT MUST BE USED SOMEHOW. THE ONLY POSSIBILITY IS: v_n IS A VERTEX ON A PATH BETWEEN TWO BRANCH VERTICES, BUT THIS PATH CAN BE SHORTENED TO NOT CONTAIN v_n.

IF WE ADD AN EDGE e IT SUFFICES TO CONSIDER ONLY THE CASE $e = v_1 v_n$ (OTHERWISE INDUCTION)

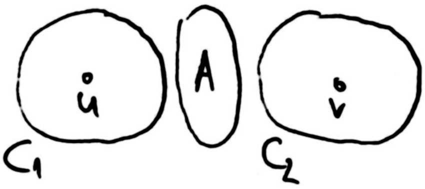


BUT IN THIS CASE APPARENTLY $K_4 \preceq_t G$. □

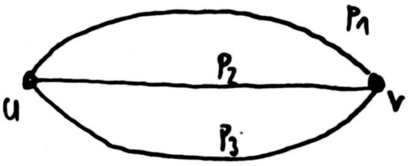
THEOREM: IF G, $|G| \geq 4$, G IS EDGE-MAXIMAL WITHOUT K_4 TOPOLOGICAL MINOR $\Rightarrow$ G IS A 2-TREE.

PROOF (INDUCTION) TRIVIAL FOR $|G| = 4$. SUPPOSE $|G| > 4$. LET $\kappa = \kappa(G)$. A BE SEPARATING SET OF VERTICES, $|A| = \kappa$. (APPARENTLY G IS NOT COMPLETE GRAPH)

1) $|A| \geq 3$

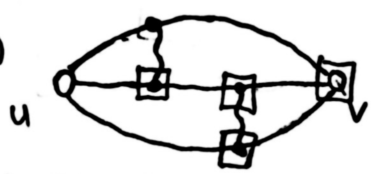


AT LEAST 3 DISJOINT u-v PATHS

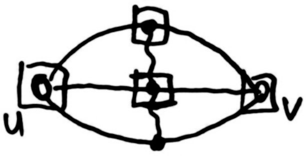


SINCE $G - \{u,v\}$ IS CONNECTED $\Rightarrow$ AT LEAST 2 PAIRS OF P_1, P_2, P_3 ARE CONNECTED BY A PATH

2 CASES: 1)

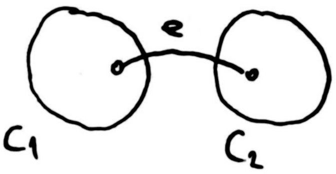


2)

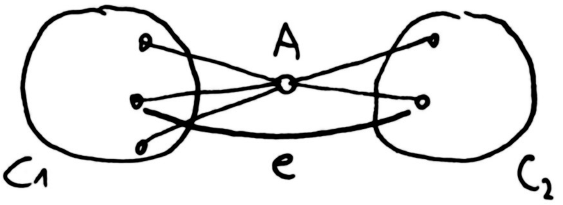


IN BOTH CASES WE HAVE FOUND K_4 TOP. MINOR

2) $|A| = 0$ WE CAN ADD AN EDGE e



$|A| = 1$



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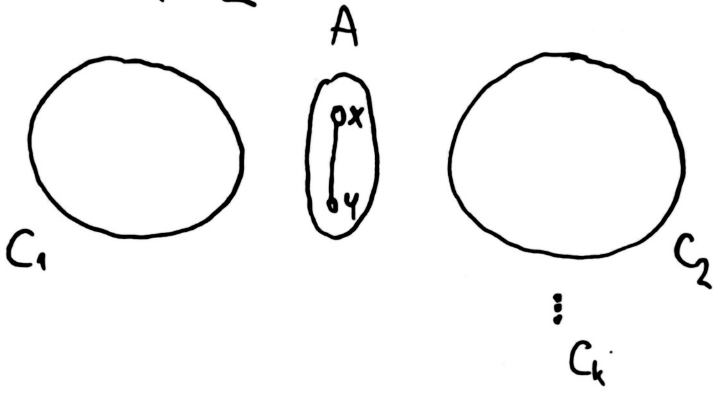
3) $|A| = 2$



WE MAY ASSUME THAT $xy \in G$, OTHERWISE WE CAN ADD THIS EDGE (x, y CANNOT BE BRANCH VERTICES OF K_4 TOP. MINOR)

DENOTE $G_i = C_i + A$.

APPARENTLY G_i IS EDGE-MAXIMAL WITHOUT K_4 MINOR. IF WE ADD AN EDGE TO G_i, K_4 TOP. MINOR WILL ARISE IN G. BUT THIS MINOR IS CONTAINED IN G_i.

$\Rightarrow$ BY INDUCTION HYPOTHESIS G_i IS A 2-TREE AND CAN BE GROWN FROM xy. $\square$

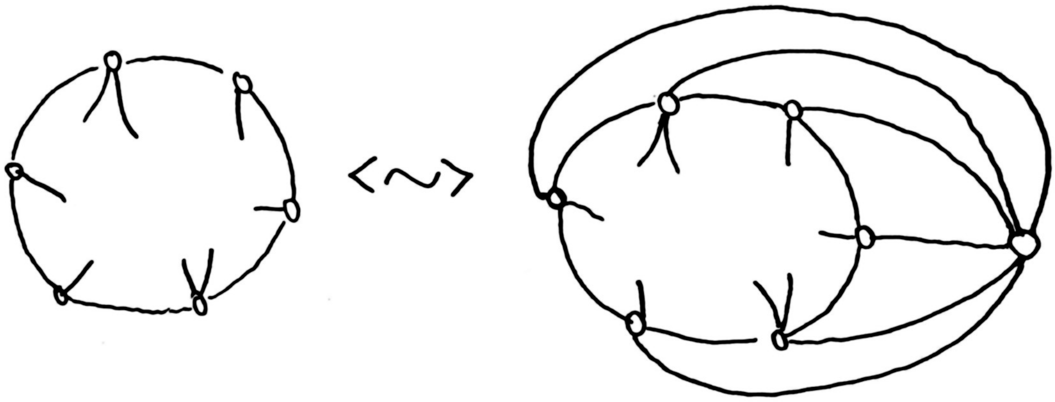
THEOREM: $\neg K_4 \preceq G \Rightarrow \neg \chi(G) \geq 4$

LET $G' \supseteq G$ BE EDGE-MAXIMAL WITHOUT K_4 MINOR ON THE SAME SET OF VERTICES.

$\Rightarrow$ G' IS A 2-TREE $\Rightarrow$ $\chi(G') \leq 3 \Rightarrow \chi(G) \leq 3$. $\square$

KURATOWSKI THEOREM FOR OUTERPLANAR GRAPHS:
G IS OUTERPLANAR $\Leftrightarrow$ G DOES NOT CONTAIN K_4, NOR $K_{2,3}$ AS A (TOPOLOGICAL) MINOR.

PROOF:



A MULTIGRAPH IS CALLED SERIES-PARALLEL IF IT CAN BE CONSTRUCTED RECURSIVELY FROM A K_2 BY THE OPERATIONS OF SUBDIVIDING AND OF DOUBLING EDGES.

2-CONNECTED (MULTIGRAPH WITHOUT K_4 (TOP.) MINOR IS SERIES-PARALLEL.

$\delta(G) \geq 3 \Rightarrow K_4 \preccurlyeq_t G$ $\square$

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NDMI012 KOMBINATORIKA A GRAFY II - SKÚŠKA

SATISFIABILITY, FORMULA, CNF, DNF,

PLAN 23.05.2008

2-CNF-SAT IS P

SAT $\leq_p$ 3-CNF-SAT

THEOREM: SATISFIABILITY OF $(k, \leq s)$-SAT

$\forall k \geq 3\ \exists f(k) : (k, \leq s)\text{-SAT} \begin{cases} \text{POLYNOMIAL FOR } s < f(k) \\ \text{NP-COMPLETE FOR } s \geq f(k) \end{cases}$

BOUNDS ON $f(k)$

LOVÁSZ LOCAL LEMMA

FORMULA Φ IN CNF (CONJUNCTIVE NORMAL FORM)

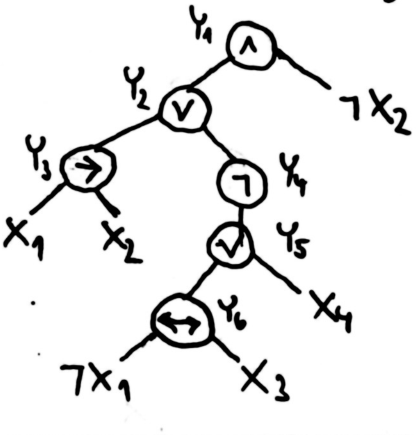
$\Phi = \bigwedge_{i=1}^{n} C_i \qquad \underbrace{C_i = \bigvee_{j=1}^{r_i} l_{ij}}_{\text{CLAUSULE}} \qquad \underbrace{l_{ij} = \begin{cases} x \\ \neg x \end{cases}}_{\text{LITERAL}}$

SAT $\leq_p$ 3-CNF-SAT

WE ARE GIVEN A FORMULA Φ AND WE WANT TO FIND A FORMULA Ψ IN 3-CNF S.T.
Φ IS SATISFIABLE $\Leftrightarrow$ Ψ IS SATISFIABLE.

FOR INSTANCE $\Phi = ((x_1 \to x_2) \vee \neg((\neg x_1 \leftrightarrow x_3) \vee x_4)) \wedge \neg x_2$

FIRST WE BUILD A TREE CORRESPONDING TO Φ AND TO EACH INNER VERTEX WE ADD NEW VARIABLE y_i



LEAVES ARE LITTERALS

Φ IS THEN EQUIVALENT TO :

$$Y_1 \wedge (Y_1 \leftrightarrow (Y_2 \wedge \neg X_2)) \wedge$$
$$\wedge (Y_2 \leftrightarrow (Y_3 \vee Y_4)) \wedge$$
$$\wedge (Y_3 \leftrightarrow (X_1 \rightarrow X_2)) \wedge$$
$$\wedge (Y_4 \leftrightarrow (\neg Y_5)) \wedge$$
$$\wedge (Y_5 \leftrightarrow (Y_6 \vee X_4)) \wedge$$
$$\wedge (Y_6 \leftrightarrow (\neg X_1 \leftrightarrow X_3)).$$

$\Phi = \bigwedge_{i=1}^{m} \varphi_i$ $\quad \varphi_i$ HAS AT MOST 3 VARIABLES.

EVERY φ_i WE CAN REPLACE WITH CONJUNCTION OF AT MOST $2^3 = 8$ CLAUSULES EACH WITH AT MOST 3 LITERALS.

THE LAST STEP IS THAT WE ADD TWO NEW VARIABLES p, q

a) $(l_1 \vee l_2)$ REPLACE WITH $(l_1 \vee l_2 \vee p) \wedge (l_1 \vee l_2 \vee \neg p)$

b) (l) REPLACE WITH $(l \vee p \vee q) \wedge (l \vee p \vee \neg q) \wedge (l \vee \neg p \vee q) \wedge (l \vee \neg p \vee \neg q)$.

$\Rightarrow$ WE GET ψ IN 3-CNF EQUIVALENT WITH Φ IN P TIME.

$(k, \leq s)$-FORMULA IS A FORMULA IN CNF, EACH CLAUSULE HAS k LITERALS AND EVERY VARIABLE OCCURS IN AT MOST s CLAUSULES.

FOR EVERY $k \geq 3$ THERE EXISTS $f(k)$ SUCH THAT $(k, \leq s)$-SAT IS $\begin{cases} P & \text{FOR } s < f(k) \\ NP & \text{FOR } s \geq f(k) \end{cases}$

APPARENTLY $f(k) \geq k+1$ SINCE EVERY $(k, \leq k)$-FORMULA IS SATISFIABLE (FROM MARRIAGE THEOREM)

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LET $A_1, \dots, A_n$ BE EVENTS. IF A_i ARE MUTUALLY DISJOINT THEN
$P(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n P(A_i) \Rightarrow P(\bigcap_{i=1}^n \bar{A}_i) > 0$
$\Leftrightarrow \sum_{i=1}^n P(A_i) < 1$.

IF A_i ARE MUTUALLY INDEPENDENT THEN
$P(\bigcap_{i=1}^n \bar{A}_i) = \prod_{i=1}^n (1 - P(A_i)) > 0 \Leftrightarrow \forall i : P(A_i) < 1$.

DEPENDENCY GRAPH OF EVENTS $A_1, \dots, A_n$ IS A GRAPH WITH VERTICES $A_1, \dots, A_n$, FOR EVERY i A_i IS INDEPENDENT TO ALL EVENTS IN $V \setminus N(A_i)$.

LÓVASZ LOCAL LEMMA: LET G BE A DEPENDENCY GRAPH OF EVENTS $A_1, \dots, A_n$ WITH MAXIMUM DEGREE d. IF FOR ALL i: $P(A_i) \leq p$ AND $pd \leq \frac{1}{4}$, THEN $P(\bigcap_{i=1}^n \bar{A}_i) > 0$.

PROOF: BY INDUCTION WE PROVE THAT
$P(A_1 \mid \bar{A}_2 \dots \bar{A}_m) \leq 2p$.

HOLDS FOR $m=1$. SUPPOSE $m>1$,
A_1 IS CONNECTED WITH $A_2, \dots, A_k$, $k \leq m$.

$P(A \mid BC) = \frac{P(AB \mid C)}{P(B \mid C)}$, SO

$$P(A_1 \mid \bar{A}_2 \dots \bar{A}_m) = \frac{P(A_1 \bar{A}_2 \dots \bar{A}_k \mid \bar{A}_{k+1} \dots \bar{A}_m)}{P(\bar{A}_2 \dots \bar{A}_k \mid \bar{A}_{k+1} \dots \bar{A}_m)}$$

NOMINATOR $\leq P(A_1 \mid \bar{A}_{k+1} \dots \bar{A}_m) = P(A_1) \leq p$

DENOMINATOR $\geq 1 - P(\bigcup_{i=2}^k A_i \mid \bar{A}_{k+1} \dots \bar{A}_m) \geq$
$\geq 1 - \sum_{i=2}^k P(A_i \mid \bar{A}_{k+1} \dots \bar{A}_m) \geq$ BY INDUCTION $\geq 1 - 2p(k-1)$

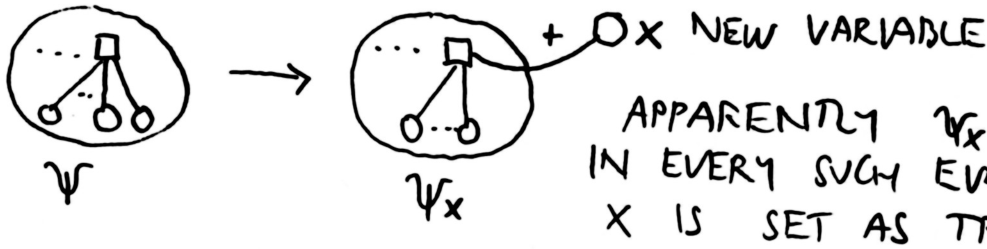
SINCE $k-1 \leq d$ AND $p \cdot d \leq \frac{1}{4} \Rightarrow 1 - 2p(k-1) \geq 1 - 2pd \geq \frac{1}{2}$.

$P(\bar{A}_1 \dots \bar{A}_n) = \prod_{i=1}^n (\bar{A}_i \mid \bar{A}_1 \dots \bar{A}_{i-1}) \geq (1-2p)^n > 0$. $\square$

LEMMA: FOR $k \geq 3$ $(k, \leq s)$-SAT IS NP

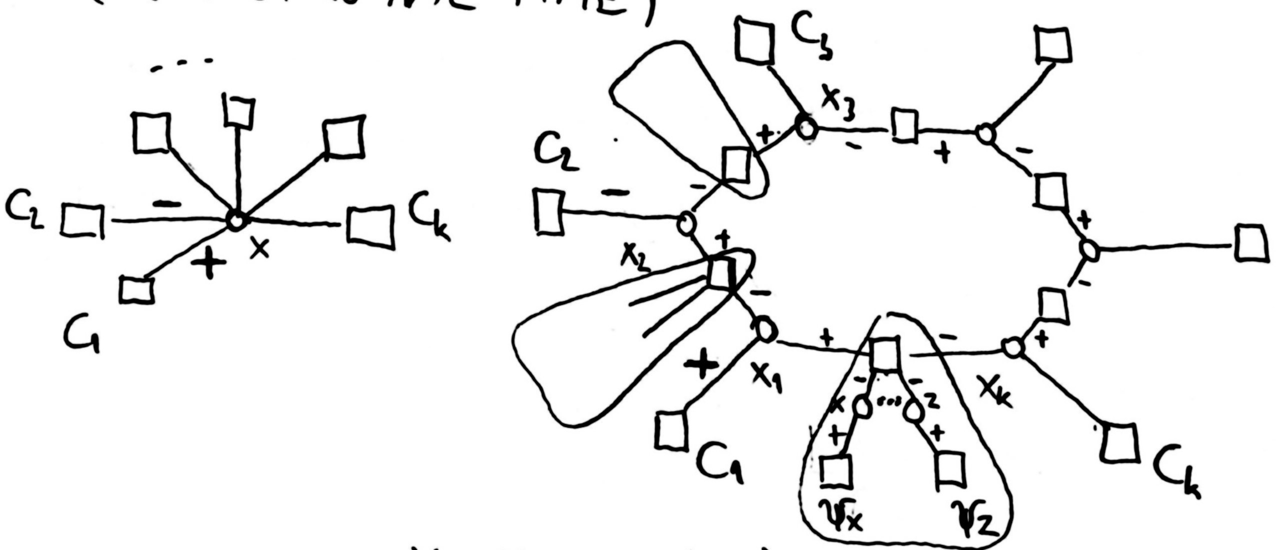
$\Leftrightarrow$ $\exists$ NONSATISFIABLE $(k, \leq s)$-FORMULA.

$\Leftarrow$) LET ψ BE MINIMAL NONSATISFIABLE $(k, \leq s)$-FORMULA.
IF WE REMOVE ANY CLAUSULE FROM ψ WE GET SATISFIABLE FORMULA.



APPARENTLY ψ_x IS SATISFIABLE IN EVERY SUCH EVALUATION X IS SET AS TRUE.

IT SUFFICES TO SHOW THAT EVERY k-CNF-SAT PROBLEM CAN BE REDUCED TO $(k, \leq s)$-SAT PROBLEM. (IN POLYNOMIAL TIME)



$$\neg X_1 \vee X_2 \equiv X_1 \Rightarrow X_2$$
$$\neg X_2 \vee X_3 \equiv X_2 \Rightarrow X_3$$
$$\cdots$$
$$\neg X_k \vee X_1 \equiv X_k \Rightarrow X_1$$

$\Big\}$ $X_1 \equiv \dots \equiv X_k$

$f(k) \leq 2^k$ TRIVIAL

THERE EXISTS A NONSATISFIABLE $(k, \leq 2^k)$-FORMULA.

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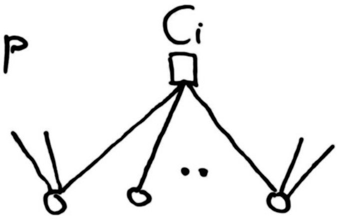
LEMMA: $\boxed{2^{k-2}/k} < f(k)$

SUPPOSE ϕ IS $(k; \leq s)$ - FORMULA
LET US EVALUATE EVERY VARIABLE X AS TRUE WITH PROBABILITY 1/2

FOR EVERY CLAUSULE C_i A BAD EVENT A_i IS THAT ALL LITERALS ARE EVALUATED AS FALSE

$P(A_i) = \left(\frac{1}{2}\right)^k = p$



EVERY CLAUSULE HAS COMMON VARIABLES WITH AT MOST $(s-1) \cdot k$ OTHER CLAUSULES. $\Rightarrow d \leq k(s-1)$

$$pd \leq \frac{1}{2^k} k(s-1) < \frac{1}{2^k} k \frac{2^{k-2}}{k} = \frac{1}{4}$$

$$\Rightarrow P\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0.$$

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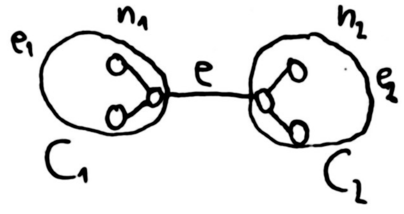
EXERCISES 09.10.2007

5) PERFECT MATCHING OF 3-REGULAR GRAPH CONTAINS ALL THE BRIDGES



IF $e \notin M \Rightarrow M$ IS PERFECT MATCHING IN G_1 AND G_2

$n = n_1 + n_2$ $\quad 3n = 2e$
$e = e_1 + e_2 + 1$ $\quad 3n_1 + 3n_2 = 2e_1 + 2e_2 + 2$
$3(n_1 - 1) + 2 = 2e_1$ $\quad 3n_1 - 1 = 2e_1$
$3(n_2 - 1) + 2 = 2e_2$ $\quad 3n_2 - 1 = 2e_2$

- BOTH C_1 AND C_2 ARE ODD
$\Rightarrow$ THEY DO NOT HAVE PERFECT MATCHING $\lightning$

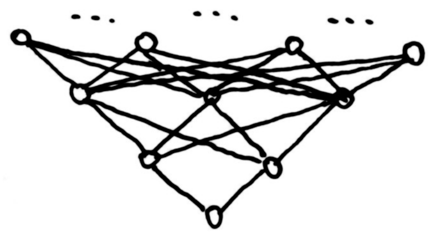
EXERCISES 16.10.2007

2) A POSET CAN BE COVERED BY AT MOST k ANTICHAINS $\Leftrightarrow$ EACH OF ITS CHAINS HAS A LENGTH AT MOST k.

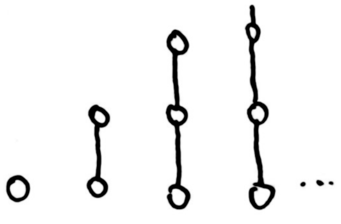
$\Rightarrow$) IF THERE IS A CHAIN OF LENGTH $> k$
- EACH ANTICHAIN CAN USE AT MOST 1 VERTEX OF THIS CHAIN $\Rightarrow$ WE NEED MORE THAN k ANTICHAINS

$\Leftarrow$) FOR EVERY $x \in X$ DEFINE $f(x) \in \{1, \dots, k\}$ THE LENGTH OF THE LONGEST CHAIN ENDING IN x.
APPARENTLY FOR EVERY $m \in \{1, \dots, k\}$ $\{x \in X : f(x) = m\}$ IS AN ANTICHAIN.

3) NO INFINITE ANTICHAIN
NO FINITE MANY COVERING CHAINS



NO INFINITE CHAIN
NO FINITE MANY COV. AN.

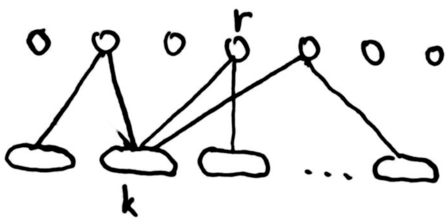


4) NASH-WILLIAMS: A MULTIGRAPH CONTAINS k EDGE-DISJOINT SPANNING TREES $\Leftrightarrow$ EVERY PARTITION P OF ITS VERTEX SET HAS AT LEAST $k(|P|-1)$ CROSS-EDGES.

EVERY $2k$-EDGE-CONNECTED GRAPH HAS k DISJ. SPAN. TR.
EACH PARTITION HAS AT LEAST $2k$ CROSS-EDGES
$2e \geq 2k \cdot |P| \Rightarrow e \geq k(|P|-1)$.

EXERCISES 23.10.2007

1) IF $r \leq k-1 \Rightarrow$ r-REGULAR k-UNIFORM HYPERGRAPH HAS A BICOLOURING.



VERTICES

EDGES

FROM MARRIAGE THEOREM $\Rightarrow$ 2 REPRESENTANTS FOR EACH EDGE

2) CONSIDER RANDOM COLOURING OF VERTICES (WITH PROB. $\frac{1}{2}$)
FOR EVERY HYPEREDGE $H_i \in E$ WE HAVE A BAD EVENT A_i, $p(A_i) = 2 \cdot \frac{1}{2^k} = 2^{1-k}$

H_i HAS COMMON VERTICES WITH AT MOST $k \cdot (r-1)$ OTHER HYPEREDGES $\Rightarrow d \leq k \cdot (r-1)$.

NOW $p \cdot d \leq \frac{k \cdot (r-1)}{2^{k-1}} < \frac{1}{4}$ FOR $r \leq \frac{2^{k-3}}{k}$

$\Rightarrow \frac{2^{k-3}}{k} \leq r_0(k)$.

CONSIDER A GRAPH WITH $2k$ VERTICES AND ALL $\binom{2k}{k}$ k-EDGES.
APPARENTLY THERE IS NO BICOLOURING
$r = \binom{2k-1}{k-1}$ i.E. $r_0(k) \leq \binom{2k-1}{k-1}$

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MENGER THEOREM: FOR $A, B \subseteq V(G)$ THE MAXIMUM NUMBER OF DISJOINT A-B PATHS IS EQUAL TO THE MINIMAL CARDINALITY OF A VERTEX SET X SEPARATING A FROM B.

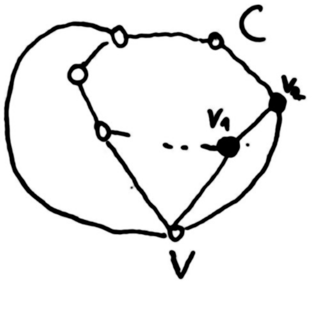
EXERCISES 30.10.2007

1) LET $|G| \geq k+1$. G IS k-CONNECTED $\Leftrightarrow$ $\forall S, T \subseteq V(G)$ $|S| = |T| = k$ THERE ARE k DISJOINT S-T PATHS.

$\Rightarrow$) LET $S, T \subseteq V(G)$, $|S| = |T| = k$. FOR EVERY $X \subseteq V(G)$, $|X| < k$ $G-X$ IS CONNECTED AND CONTAINS AT LEAST ONE VERTEX FROM S AND ONE FROM T.

$\Leftarrow$) SUPPOSE THERE IS A SET $X \subseteq V(G)$ $|X| < k$ S.T. $G-X$ IS DISCONNECTED, u, v ARE FROM DIFFERENT COMPONENTS.
LET $S, T \subseteq V(G)$, $|S| = |T| = k$, $u \in S$, $u \notin T$, $v \in T$, $v \notin S$.
IF THERE ARE k DISJOINT S-T PATHS $\Rightarrow$ S CANNOT BE SEPARATED FROM T BY X. $\lightning$.

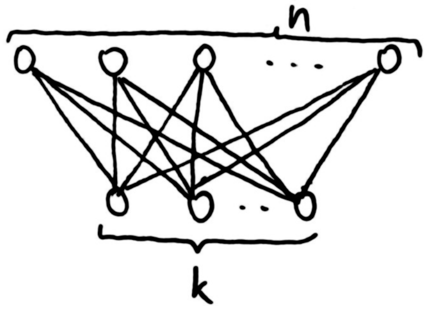
2) G k-CONNECTED GRAPH WITH AT LEAST $2k$ VERTICES
LET C BE THE LONGEST CYCLE IN G. SUPPOSE $|C| < 2k$.
LET $v \in G - C$. SINCE $d(v) \geq \delta(G) \geq \lambda(G) \geq \kappa(G) = k$ AND $|C| \geq \delta(G) + 1 \geq \kappa(G) + 1 = k+1$
THERE ARE AT LEAST k DISJOINT $N(v) - V(C)$ PATHS I.E. k $v - V(C)$ FAN.



SINCE $|C| < 2k$ THERE ARE TWO NEIGHBOURS $v_1, v_k \in C$ CONNECTED WITH v $\Rightarrow$ C CAN BE PROLONGED $\lightning$.

THERE ARE ARBITRARILY LARGE k-CONNECTED GRAPHS WITH NO CYCLE OF LENGTH $> 2k$.

CONSIDER BIPARTITE GRAPH



3) EVERY k-LINKED GRAPH IS $(2k-1)$-CONNECTED

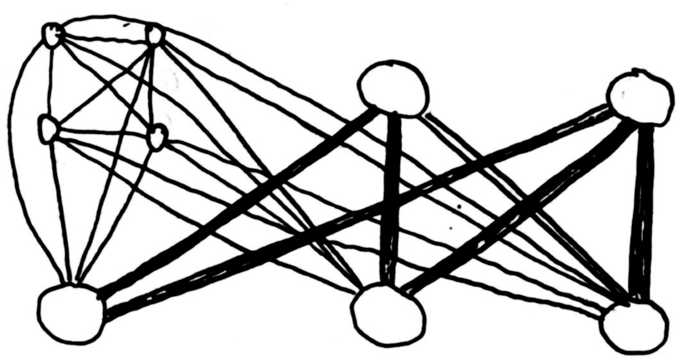
SUPPOSE G IS k-LINKED, I.E. $|G| \geq 2k$
AND $X \subseteq V(G)$, $|X| < 2k-1$ SEPARATES 2 VERTICES u, v.

IT IS NOT POSSIBLE TO CONNECT u_i WITH v_i AND u WITH v

WE MAY ASSUME THAT $|X| = 2k-2$.

$G = K_{2k-1, 2k-1}$ IS k-LINKED, BUT NOT $2k$-CONNECTED, SINCE $\delta(G) = 2k-1 \geq \lambda(G) \geq \kappa(G)$.

WE MAY REPLACE ONE VERTEX OF $K_{2k-1,2k-1}$ WITH K^n GRAPHS $n \geq 2k-1$



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EXERCISES 06.11.2007

1) SUPPOSE A HAS EXACTLY $k+s-2$ ELEMENTS. THEN $\binom{A}{s}$ HAS NO k - SUNFLOWER.

IF THE CORE HAS r ELEMENTS $0 \le r \le s-1$

CORE $\quad$ k DISJOINT PARTS

r $\quad$ $s-r$ VERTICES $\quad$ $\Rightarrow \underline{\underline{s-r \geq 1}}$

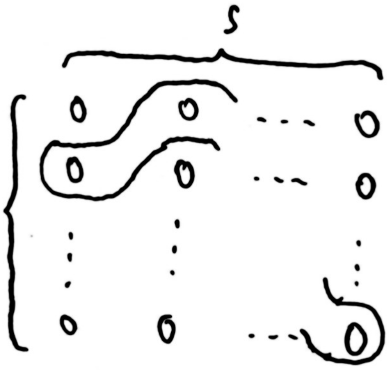
TOGETHER $r + k(s-r) > k+s-2$

HOLDS FOR $k=1$

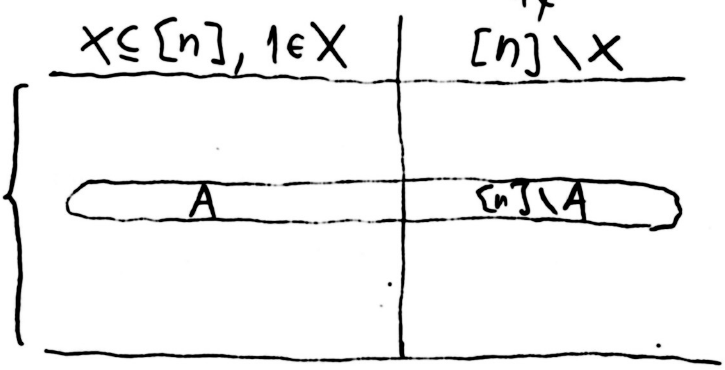
FOR $k \geq 1$ $\quad (k-1)(s-r-1) - 1 > -2$

$(k-1)(s-r-1) \geq 0$. $\square$

2)



3) CONSIDER TABLE



$\mathcal{P}(X)$ PARTITIONED INTO 2 COLUMNS

IF $|\mathcal{F}| > 2^{n-1}$
$\Rightarrow$ THERE IS A ROW S. T. BOTH SETS ARE SELECTED ↯
(PIGEONHOLE PRINCIPLE

4) $k \geq 2$, $G=(V,E)$, $|E| \geq (k-1)|V| \Rightarrow \varepsilon(G) = \frac{|E|}{|V|} \geq k-1$

TREE WITH k EDGES HAS $k+1$ VERTICES

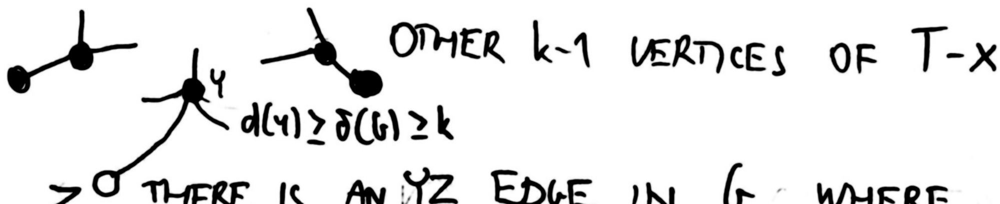
THERE IS A SUBGRAPH $H \subseteq G$ S.T. $\delta(H) > \varepsilon(H) \geq \varepsilon(G)$
I.E. $\delta(H) \geq k$.

IT SUFFICES TO SHOW THAT EVERY TREE WITH k EDGES IS CONTAINED IN A GRAPH WITH $\delta(G) \geq k$.

FOR $k=1$ TRIVIALLY HOLDS

SUPPOSE $k > 1$. IF WE DELETE A LEAF$_x$ FROM OUR TREE T
BY INDUCTION $T-x$ IS CONTAINED IN G
SUPPOSE THAT x SHOULD BE CONNECTED TO y IN G.

SINCE y IN $T-x$ HAS DEGREE AT MOST $k-1$ (# OF EDGES)
AND $\delta(G) \geq k$ $\Rightarrow$



THERE IS AN YZ EDGE IN G WHERE
$z \notin T-x$.

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EXERCISES 13.11.2007

1) CHOOSABILITY OF $\boxed{K^n - e}$

COLOURING NUMBER OF G

FOR EVERY GRAPH G : $ch(G) \leq \underbrace{col}(G) = \max_{H \subseteq G} \delta(H) + 1$

$col(K^n - e) = n-2$ AND APPARENTLY $\chi(K^n - e) = n-2$

$\Rightarrow$ SINCE $\chi(G) \leq ch(G)$ $\Rightarrow$ $ch(K^n - e) = n-2$.

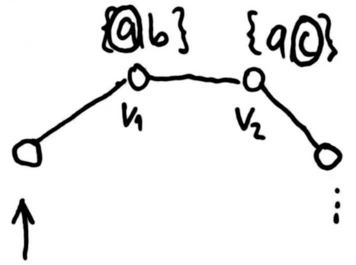
$\boxed{ch(C^n)}$: FOR ODD n $ch(C^n) = 3$.

BECAUSE $3 = \chi(C^n) \leq ch(C^n) \leq col(C^n) = 3$.

SUPPOSE n IS EVEN. LET US PROVE THAT $ch(C^n) = 2$.

a) IF THERE ARE 2 ADJACENT VERTICES v_1, v_2, WHOSE LISTS HAVE AT MOST 1 COLOUR IN COMMON

$\Rightarrow$ WE CHOOSE ONE COMBINATION

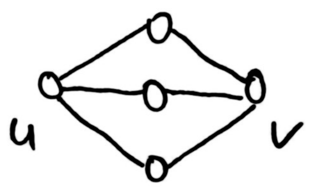


WE PROCEED ALGORITMICALY

IF WE COLOUR THIS VERTEX WITH a IN THE END, WE JUST CHANGE COLOUR OF v_1 TO b.

b) ALL LIST ARE SAME $\Rightarrow$ THERE IS A BICOLOURING.

$\boxed{ch(K_{2,3})}$



APPARENTLY $ch(K_{2,3}) = 2$

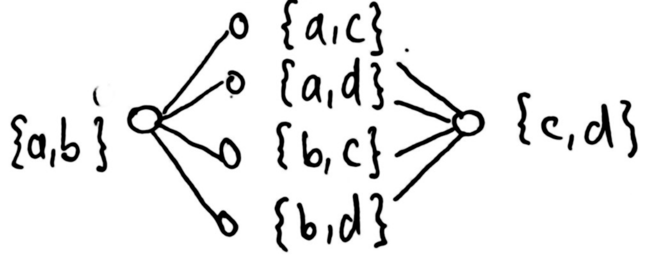
TWO CASES: a) $L(u) \cap L(v) \neq \emptyset$ b) $L(u) \cap L(v) = \emptyset$.

$\boxed{ch(K_{2,4}) = 3}$

$\Rightarrow$ FOR $n \geq 4$ $ch(K_{2,n}) = 3$

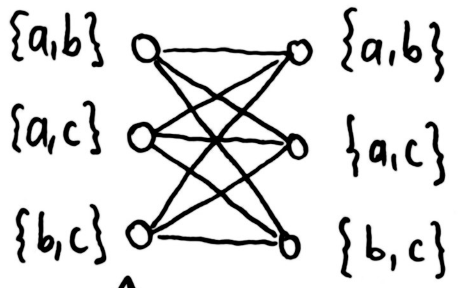
SINCE $K_{2,4} \subseteq K_{2,n}$

AND $col(K_{2,n}) = 3$



$\boxed{ch(K_{3,3})}$

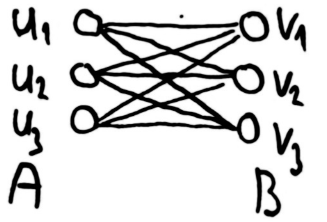
$\chi(K_{3,3}) = 2 \leq ch(K_{3,3}) \leq col(K_{3,3}) = 4.$



AT LEAST 2 COLOURS

$\Rightarrow ch(K_{3,3}) \geq 3$

LET US PROVE THAT $ch(K_{3,3}) \leq 3$.



IF A (OR B) HAS AT LEAST 1 COMMON COLOUR $\Rightarrow$ IT IS EASY

SUPPOSE THAT LISTS IN A (AND B) ARE DISJOINT.

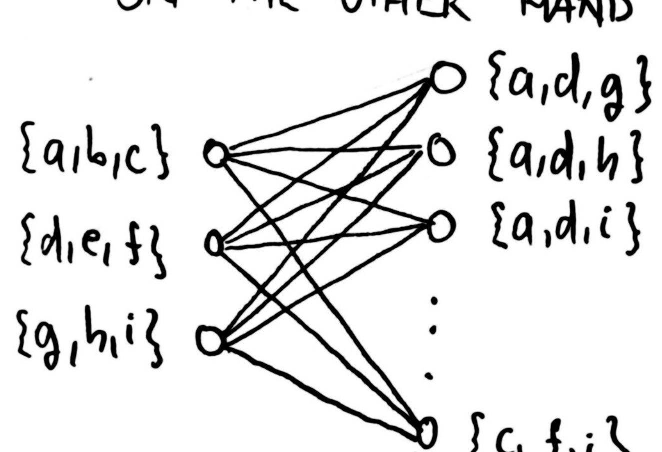
WE CHOOSE $f(u_1), f(u_2), f(u_3)$ S.T.

$\{f(u_1), f(u_2), f(u_3)\} \neq L(v_i)$ FOR $i = 1,2,3$.

$\boxed{ch(K_{3,27})}$

$ch(K_{3,27}) \leq col(K_{3,27}) = 4.$

ON THE OTHER HAND:



3 $\qquad$ $3^3 = 27$

1/5

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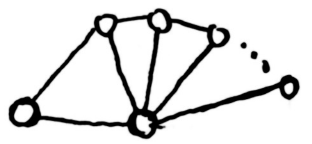
EXERCISES 27.11.2007

3) $tw(K^n) = n-1$

SINCE K^n IS A $(n-1)$-TREE
AND EVERY k-TREE HAS A VERTEX OF DEGREE k.

$tw(C^n) = 2$ SINCE PARTIALLY 1-TREES ARE FORESTS

AND

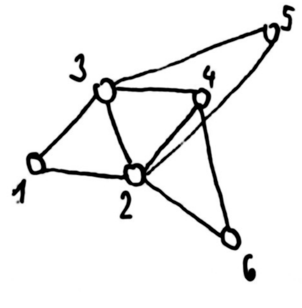


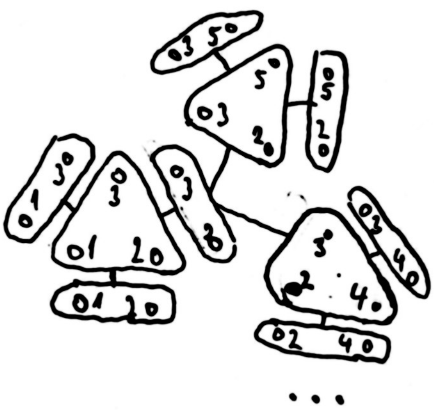
SO THERE IS ALWAYS
A 2-TREE $\supseteq C^n$.

SUPPOSE $m \leq n$. $\delta(K_{m,n}) = m \Rightarrow tw(K_{m,n}) \geq m$.
BUT IT IS OBVIOUS THAT $tw(K_{m,n}) = m$ $(\min\{m,n\})$.

4) THERE ARE AT LEAST TWO NONADJACENT VERTICES OF DEGREE AT MOST k IN EVERY (PARTIAL) k-TREE WITH MORE THAN $k+1$ VERTICES.

5)





...

EXERCISES 04.12.2007

SZEMERÉDI'S REGULARITY LEMMA:

FOR EVERY $\varepsilon > 0$, $m \geq 1$ $\exists M \in \mathbb{N}$ S.T. EVERY GRAPH
□ OF ORDER AT LEAST m ADMITS ε-REGULAR PARTITION
$\{V_0, V_1, \dots, V_k\}$ WITH $m \leq k \leq M$.

○ FOR EVERY ... $\exists M \in \mathbb{N}$ $\exists n_0 \in \mathbb{N}$ S.T. ...
OF ORDER AT LEAST n_0

1) □ $\Rightarrow$ ○) IT SUFFICES TO SET $n_0 := m$.
○ $\Rightarrow$ □) SUPPOSE $n_0 > m$
WE MAY ASSUME $M \geq n_0$, BECAUSE THE THEOREM HOLDS IF WE INCREASE M.
BUT FOR G WITH $m \leq |G| < n_0 \leq M$
THERE IS ALWAYS ε-REGULAR PARTITION OF G INTO $|G|$ SINGLETON SETS.

2) $\forall \varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ $\exists \delta > 0$: IF (A, B) IS AN ε-REGULAR PAIR WITH $|A| = |B| = n \geq n_0$,
THEN IF WE ADD TO PARTITIONS A, B
AT MOST δn VERTICES AND SOME EDGES,
WE GET 2ε-REGULAR PAIR.

SUPPOSE THAT (A, B) IS ε-REGULAR, $|A| = |B| = n$,
AND WE ADD A' TO A, B' TO B, $|A'| + |B'| < \delta n$
AND SOME NEW EDGES FROM A' (OR B').

FOR EVERY $X \subseteq A$, $Y \subseteq B$, $|X| \geq \varepsilon |A|$, $|Y| \geq \varepsilon |B|$
$|d(X, Y) - d(A, B)| \leq \varepsilon$.

25.05.2008 Su
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NDMI012 KOMBINATORIKA A GRAFY II - SKÚŠKA

LET $d = d(A,B) = \frac{\|A,B\|}{|A|\cdot|B|}$, $\tilde{d} = d(A\cup A', B\cup B') = \frac{\|A\cup A', B\cup B'\|}{|A\cup A'|\cdot|B\cup B'|}$

$n = |A| \leq |A\cup A'| \leq (1+\delta)n$

$n = |B| \leq |B\cup B'| \leq (1+\delta)n$

$$\tilde{d} \leq \frac{\|A,B\| + \|A',B\cup B'\| + \|A\cup A',B'\|}{|A\cup A'|\cdot|B\cup B'|} \leq$$

$$\leq \frac{\|A,B\|}{|A|\cdot|B|} + \frac{\|A',B\cup B'\| + \|A\cup A',B'\|}{|A\cup A'|\cdot|B\cup B'|} \leq$$

$$\leq d + \frac{\delta n\cdot(1+\delta)n + \delta n(1+\delta)n}{n^2} =$$

$$= d + 2\delta(1+\delta) < d + \varepsilon$$

$$\tilde{d} \geq \frac{\|A,B\|}{|A\cup A'|\cdot|B\cup B'|} \geq \frac{\|A,B\|}{(1+\delta)^2|A|\cdot|B|} = \frac{1}{(1+\delta)^2} d > d - \varepsilon$$

SUPPOSE THAT $X \subseteq A$, $X' \subseteq A'$, $Y \subseteq B$, $Y' \subseteq B'$

$|X\cup X'| \geq 2\varepsilon|A\cup A'|$, $|Y\cup Y'| \geq 2\varepsilon|B\cup B'|$

WE REQUIRE THAT FROM THESE INEQUALITIES FOLLOWS THAT $|X| \geq \varepsilon|A|$ AND $|Y| \geq \varepsilon|B|$.

IF $|X| < \varepsilon|A|$, THEN $X\cup X' \subseteq X\cup A'$

$\Rightarrow |X\cup X'| \leq |X\cup A'| = |X| + |A'| < \varepsilon|A| + |A'| <$

$< \varepsilon n + \delta n = (\varepsilon+\delta)n < 2\varepsilon|A| < 2\varepsilon|A\cup A'|$

FOR $\delta < \varepsilon$.

$$d(X\cup X', Y\cup Y') = \frac{\|X\cup X', Y\cup Y'\|}{|X\cup X'|\cdot|Y\cup Y'|} \leq$$

$$\leq \frac{\|X,Y\| + \|X', Y\cup Y'\| + \|X\cup X', Y'\|}{|X\cup X'|\cdot|Y\cup Y'|} \leq$$

$$\leq d(X,Y) + \frac{\|X', Y\cup Y'\| + \|X\cup X', Y'\|}{|X\cup X'|\cdot|Y\cup Y'|} \leq$$

$$\leq d(X,Y) + \frac{\|A', B\cup B'\| + \|A\cup A', B'\|}{4\varepsilon^2|A\cup A'|\cdot|B\cup B'|} \leq$$

$$\leq d(X,Y) + \frac{1}{2}\frac{\delta(1+\delta)}{\varepsilon^2}. < d(X,Y)+\varepsilon$$

$$d(X\cup X', Y\cup Y') \geq \frac{\|X,Y\|}{|X\cup X'|\cdot|Y\cup Y'|} \geq$$

$$\geq \frac{\|X,Y\|}{|X\cup A'|\cdot|Y\cup B'|} \geq \qquad |X\cup A'| = |X| + |A'| < |X| + \delta n$$

$$\geq \frac{\|X,Y\|}{(|X|+\delta n)(|Y|+\delta n)} \geq \qquad \frac{\delta}{\varepsilon}|X| \geq \delta n$$

$$\Rightarrow |X| + \delta n \leq |X|\left(1+\frac{\delta}{\varepsilon}\right)$$

$$\geq \frac{\|X,Y\|}{|X|\cdot|Y|\cdot\left(1+\frac{\delta}{\varepsilon}\right)^2} =$$

$$= \frac{1}{\left(1+\frac{\delta}{\varepsilon}\right)^2} d(X,Y) > d(X,Y) - \varepsilon$$

$$\boxed{d(X,Y)\left(1-\frac{1}{\left(1+\frac{\delta}{\varepsilon}\right)^2}\right) < (d+\varepsilon)\left(1-\frac{1}{\left(1+\frac{\delta}{\varepsilon}\right)^2}\right) < \varepsilon}$$

$\Rightarrow$ WE HAVE TO CHOOSE $0<\delta<\varepsilon$ S.T.

$2\delta(1+\delta) < \varepsilon$, $d\left(1-\frac{1}{(1+\delta)^2}\right) < \varepsilon$, $\frac{1}{2}\frac{\delta(1+\delta)}{\varepsilon^2} < \varepsilon$,

$(d+\varepsilon)\left(1-\frac{1}{\left(1+\frac{\delta}{\varepsilon}\right)^2}\right) < \varepsilon$ AND n_0 S.T. $\delta n_0 \geq 1$.

25.05.2008 Su
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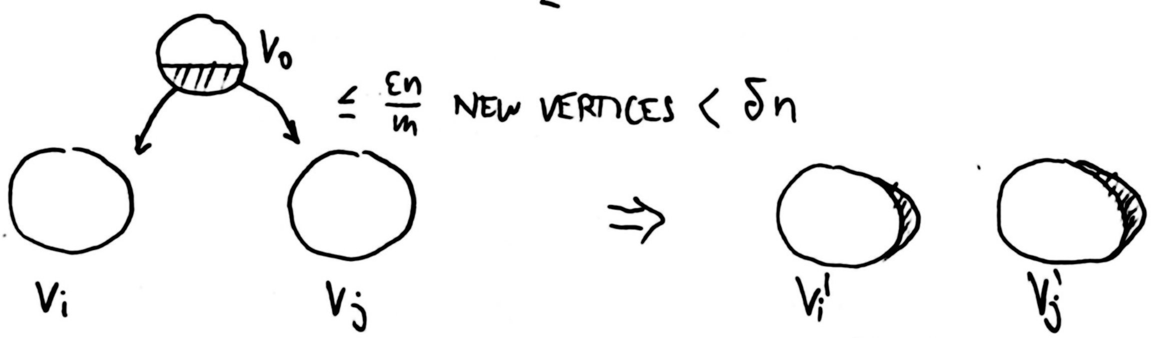
NDMI012 KOMBINATORIKA A GRAFY II - SKÚŠKA

WE ARE GIVEN $\vartheta > 0$ AND m AND WE WANT TO FIND M AND n_0 S.T. EVERY GRAPH WITH AT LEAST n_0 VERTICES HAS ϑ-REGULAR* PARTITION.

FROM OUR LEMMA IF WE SET $\varepsilon := \frac{\vartheta}{2}$ WE GET δ S.T. IF WE ADD TO ANY ε-REGULAR PAIR AT MOST δn VERTICES, WE GET 2ε-REGULAR PAIR.

FOR SOME $\varepsilon > 0$ AND m WE GET M FROM REGULARITY LEMMA
THE EXCEPTIONAL SET HAS SIZE AT MOST εn
SO V_i $i \geq 1$ CAN OBTAIN AT MOST $\frac{\varepsilon n}{k} \leq \left\lceil \frac{\varepsilon n}{m} \right\rceil$ NEW VERTICES. WE HAVE TO CHOOSE n_0 AND ε S.T. $\delta n > 2\left\lceil \frac{\varepsilon n}{m} \right\rceil$ AND $\varepsilon < \frac{\vartheta}{2}$. FOR ALL $n \geq n_0$



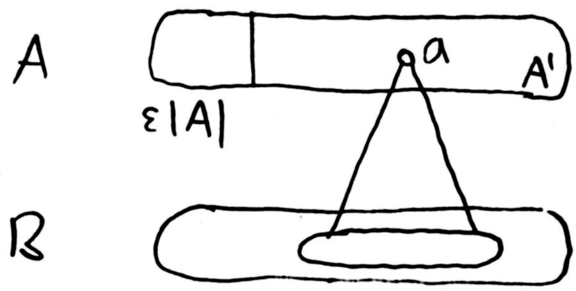
ε-REGULAR PAIR

2ε-REGULAR PAIR
$2\varepsilon < \vartheta$

3) LET $d \in (0,1\rangle$ (A,B) IS AN ε-REGULAR PAIR FOR ε SUFFICIENTLY SMALL WITH RESPECT TO d.
$\#(a,a')$: $a, a' \in A$, $|N(a) \cap N(a')| < (d-\varepsilon)^2|B|$ IS AT MOST $2\varepsilon|A|^2$.

WE KNOW THAT FOR EVERY $Y \subseteq B$, $|Y| \geq \varepsilon|B|$ THERE ARE AT MOST $\varepsilon|A|$ VERTICES IN A WITH $< (d-\varepsilon)|Y|$ NEIGHBOURS IN Y.

$\Rightarrow$ THERE ARE AT MOST $\varepsilon|A|$ VERTICES IN A WITH DEGREE $< (d-\varepsilon)|B|$ ... THE OTHERS LET BE IN $A' \subseteq A$



AT LEAST $(d-\varepsilon)|B|$ NEIGHBOURS

$(d-\varepsilon)|B| \geq \varepsilon|B|$
$\Rightarrow \boxed{\varepsilon \leq \frac{d}{2}}$

LET $Y := N(a)$, WHERE $a \in A'$.
IN A THERE ARE AT MOST $\varepsilon|A|$ VERTICES WITH $< (d-\varepsilon)|Y|$ NEIGHBOURS IN Y.

$\Rightarrow$ FOR ANY $a \in A'$ THERE ARE AT LEAST $(1-\varepsilon)|A|$ VERTICES WITH AT LEAST $(d-\varepsilon)|N(a)| \geq (d-\varepsilon)^2|B|$ NEIGHBOURS IN $N(a)$.

$\Rightarrow$ SINCE $|A'| \geq (1-\varepsilon)|A|$ WE HAVE AT LEAST $(1-\varepsilon)^2|A|^2$ CORRECT PAIRS
$\Rightarrow |A|^2 - (1-\varepsilon)^2|A|^2 = |A|^2(1-1+2\varepsilon-\varepsilon^2) < 2\varepsilon|A|^2$.

25.05.2008 Su
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NDMI012 KOMBINATORIKA A GRAFY II - SKÚŠKA

SUMMARY OF EXTREMAL GRAPH THEOREMS:

TURÁN THEOREM: FOR EVERY $r \geq 2$ EVERY GRAPH $G \not\supseteq K^r$ ON n VERTICES WITH $ex(n, K^r)$ EDGES IS $T^{r-1}(n)$.

$$t_{r-1}(n) \leq \frac{1}{2} n^2 \frac{r-2}{r-1} \quad \text{WITH EQUALITY IF } r-1 \mid n$$

$$\lim_{n \to \infty} t_{r-1}(n) \binom{n}{2}^{-1} = \frac{r-2}{r-1}$$

ERDÖS-STONE THEOREM: FOR EVERY $r \geq 2$, $s \geq 1$ $\gamma > 0$ $\exists n_0 \in \mathbb{N}$ S.T. EVERY GRAPH ON $n \geq n_0$ VERTICES WITH AT LEAST $t_{r-1}(n) + \gamma n^2$ EDGES CONTAINS K_s^r AS A SUBGRAPH

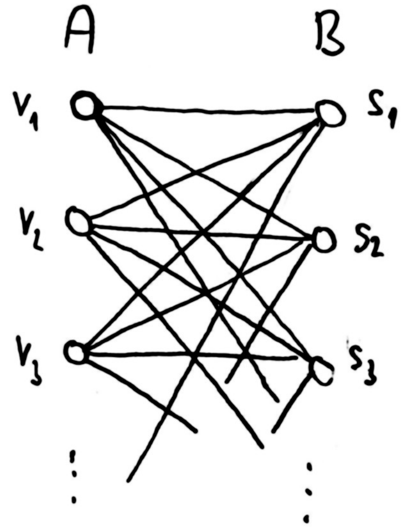
COROLLARY: FOR EVERY GRAPH H WITH AT LEAST ONE EDGE: $\lim_{n \to \infty} ex(n, H) \binom{n}{2}^{-1} = \frac{\chi(H)-2}{\chi(H)-1}$.

SZEMERÉDI REGULARITY LEMMA: FOR EVERY $\varepsilon > 0$, $m \geq 1$ $\exists M$: EVERY GRAPH WITH AT LEAST m VERTICES HAS AN ε-REGULAR PARTITION $\{V_0, \dots, V_k\}$, $m \leq k \leq M$.

LEMMA: FOR EVERY $d \in (0;1\rangle$, $\Delta \geq 1$ $\exists \varepsilon_0 > 0$ S.T. FOR ANY GRAPH G, ANY GRAPH H WITH $\Delta(H) \leq \Delta$, ANY $s \in \mathbb{N}$ IF R IS A REGULARITY GRAPH OF G WITH PARAMETERS $\varepsilon \leq \varepsilon_0$, $\ell \geq \frac{s}{\varepsilon_0}$, d, THEN $H \subseteq R_s \Rightarrow H \subseteq G$.

EXERCISES 18.12.2007

2)



THERE IS A COLOUR c_1 SUCH THAT INFINITELY MANY EDGES FROM v_1 HAVE THIS COLOUR c_1. LET $I_1 \subseteq B$ IS THE SET OF THE ENDPOINTS OF THESE EDGES IN B. I_1 IS INFINITE.

SUPPOSE THAT COLOURS $c_1, \dots, c_i$ AND SETS $V_1 \supseteq \dots \supseteq V_i$ ARE DEFINED FOR VERTICES $c_1, \dots, c_i$.

SINCE V_i IS INFINITE THERE IS A COLOUR c_{i+1} SUCH THAT INFINITELY MANY EDGES FROM v_{i+1} TO V_i HAVE THE COLOUR c_{i+1}.
LET $V_{i+1} \subseteq V_i$ BE THE SET OF ENDPOINTS OF SUCH EDGES.

IF WE USE k COLOURS THEN FOR ANY $n \in \mathbb{N}$ BETWEEN VERTICES $v_1, \dots, v_{kn+1}$ THERE ARE AT LEAST n VERTICES OF THE SAME COLOUR.

SUPPOSE $v_{i_1}, \dots, v_{i_n}$ $\quad i_1 < \dots < i_n$ HAVE COLOUR c.

THEN $\{v_{i_1}, \dots, v_{i_n}\} - V_{i_n}$

IS A MONOCHROMATIC $K_{n,\infty}$ SUBGRAPH.

LET US COLOUR EDGES $v_i - s_j$ OF $K_{A,B}$: $c(v_i s_j) = \begin{cases} \text{BLUE} & i \leq j \\ \text{RED} & i > j \end{cases}$

$\Rightarrow$ $K_{A,B}$ DOES NOT CONTAIN MONOCHROMATIC $K_{\infty,\infty}$.

25.05.2008 Su
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NDMI012 KOMBINATORIKA A GRAFY II - SKÚŠKA

⚠ REMOVAL LEMMA, PROOF OF ERDÖS-STONE !

EXERCISES 08.01.2008

THEOREM ([EC]: 9.1(98)): SUPPOSE THAT THE LARGEST CHAIN IN THE POSET P HAS SIZE r. THEN P CAN BE PARTITIONED INTO r ANTICHAINS.

PROOF: $f: X \to \{1, \dots, r\}$ $f(x)$ = LENGTH OF THE LONGEST CHAIN ENDING WITH x.
$A_i = \{x \in X : f(x) = i\}$. APPARENTLY $\{A_1, \dots, A_r\}$ IS PARTITION OF X INTO r ANTICHAINS.

2) DILWORTH'S DECOMPOSITION THEOREM ([EC]: 9.2(98)): SUPPOSE THAT THE LARGEST ANTICHAIN IN THE POSET P HAS SIZE r. THEN P CAN BE PARTITIONED INTO r CHAINS. (FOR COUNTABLE P)

SUPPOSE $P = \{x_1, x_2, \dots, x_r, \dots\}$ WHERE $\{x_1, \dots, x_r\}$ IS AN ANTICHAIN.
FOR EVERY $i \geq r$ $P_i = \{x_1, \dots, x_i\}$ CAN BE COVERED WITH r CHAINS $C_1^{(i)}, \dots, C_r^{(i)}$, WE MAY SUPPOSE THAT $x_j \in C_j^{(i)}$ FOR $i = 1, \dots, r$.
FOR EVERY x_k WE CAN DEFINE $f: X \to \{1, \dots, r\}$ AS $f(x_k) = j \Leftrightarrow$ FOR INFINITELY MANY $i \geq k$ IN $I^{(k-1)}$ IN PARTITIONING OF P_i, $x_k \in C_j^{(i)}$. (FOR $k \leq r$ $f(x_k) = k$) → DEFINE $I^{(k)} \subseteq I^{(k-1)}$
LET $C_j = \{x \in X : f(x) = j\}$ $j = 1, \dots, r$

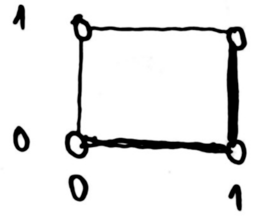
APPARENTLY $C_1, \dots, C_r$ ARE CHAINS COVERING P.

3) $HJ(k,2) \leq k$

WE WANT TO SHOW THAT ANY
k-COLOURING $c: \{0,1\}^k \to \{1,\dots,k\}$ OF $\{0,1\}^k$
HAS A COMBINATORIAL LINE

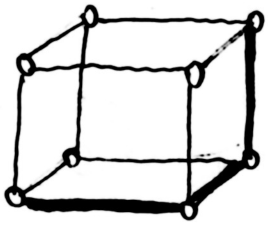
FOR $k=1$ TRIVIAL

$k=2$



HOLDS

$k=3$



LET US TAKE A SEQUENCE OF WORDS $\in \{0,1\}^k$:

$$\left.\begin{array}{l} 00\dots0 \\ 00\dots1 \\ \dots\dots\dots \\ 01\dots1 \\ \underbrace{11\dots1}_{k} \end{array}\right\} k+1 \text{ WORDS}$$

$\Rightarrow$ 2 OF THEM HAVE THE SAME COLOUR !

$$\left.\begin{array}{l} 00\,00\,111 \\ 00\,11\,111 \end{array}\right\} \; 00 * * 111 \text{ IS A COMBINATORIAL LINE}$$

4) CONSIDER $\varepsilon = \frac{1}{k}$. FOR $N = N(\varepsilon, \ell)$, $A = \{0,\dots,\ell-1\}$
LET c BE ANY COLOURING OF A^N WITH k COLOURS
$\Rightarrow$ THERE IS A COLOUR c SUCH THAT $X \subseteq A^N$ WITH THIS COLOUR CONTAINS AT LEAST $\frac{1}{k}\ell^N =: \varepsilon \ell^N$ VECTORS
$\Rightarrow$ HAS A COMBINATORIAL LINE.