Poznámky z přednášek
Univerzita Karlova v Praze
Matematicko-fyzikální fakulta

Kombinatorika a grafy II [1]

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[1] v současnosti Kombinatorika a grafy II a III

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Anotace: Přehledová přednáška o klasických výsledcích v kombinatorice a teorii grafů. Přednáška je určena primárně pro studenty magisterského stupně, popř. studenty prvního ročníku doktorského studia.

Sylabus:

1. Klasické výsledky z teorie grafů (Brooksova a Vizingova věta, Tutteho věta a její algoritmický důkaz).
2. Klasické výsledky z exremální kombinatoriky (Turanova věta, Ramseyova teorie), příklady netradičních důkazových technik (pravděpodobnostní metoda).
3. Nové trendy v teorii grafů (vybíravost grafu, Szemeredi Regularity Lemma, Removal Lemma), pokročilé výsledky z extremální kombinatoriky (Hales- Jewettova věta, Gallai-Wittova věta).
4. Přednáška probíhá v angličtině, cvičení česky.

Literatura:

1. R. Diestel, Graph theory, 3rd edition, Springer, 2005.
2. S. Jukna, Extremal combinatorics with application in computer science, Springer, 2001.

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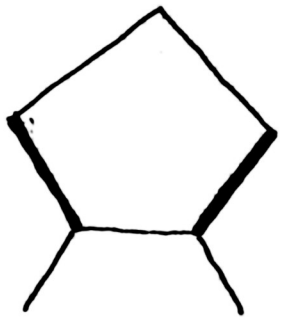
DM1012 KOMB. A GRAFY II

MAXIMUM PERFECT MATCHING

DEF.: $M \subseteq E_G$ IS A MATCHING IF
$\forall e \neq f \in M: e \cap f = \emptyset$
M IS PERFECT IF $|M| = \frac{|V_G|}{2}$
(IF $\forall u \in V_G \ \exists e \in M: u \in e$)
M IS MAXIMAL IF WE CANNOT ADD ANY EDGE
M IS MAXIMUM MATCHING IF $|M| = \max |N|$
N IS MATCHING



IS MAXIMAL, BUT
IS NOT MAXIMUM MATCHING

THERE IS NO PERFECT
MATCHING IN THIS GRAPH

THEOREM (TUTTE): G HAS A PERFECT MATCHING
$\Leftrightarrow$ (TC) $\forall A \subseteq V_G \quad \underbrace{c_{odd}(G \setminus A)}_{\text{NUMBER OF COMPONENTS WITH ODD NUMBER OF VERTICES}} \leq |A|$

PROOF: $\Rightarrow$

APPARENTLY $|A| \geq c_{odd}(G \setminus A)$

$\Leftarrow$) 1) OBSERVATION : (TC) $\Rightarrow$ $2 \mid |V_G|$

FOR $A = \emptyset$ $\quad c_{odd}(G) \leq |\emptyset| = 0$.

INDUCTION ON THE NUMBER OF NONEDGES ON A FIXED VERTEX SET

1. $G \cong K_n$, n IS EVEN $\Rightarrow$ OK ✓
2. $G \ncong K_n$

WE ASSUME $\forall G' \supseteq G$ THE STATEMENT IS TRUE

$\forall e \in \binom{V_G}{2} \setminus E_G$ $\quad G+e$ IS OK

2) OBSERVATION : IF G SATISFIES TC $\Rightarrow$ $G+e$ SATISFIES TC

3. $\Rightarrow$ $G+e$ HAS A PM

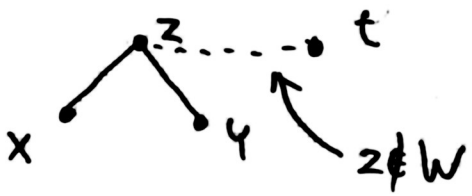
4. $W = \{u \in V_G \mid \forall x \in V_G - u \quad xu \in E_G\}$

5.

a) $G \setminus W$ IS DISJOINT UNION OF COMPLETE SUBGRAPHS $\Rightarrow$ OK

b) $G \setminus W$ IS NOT ...

$\Rightarrow$ $\exists$



IN $V_G - W$

$\Rightarrow$ $G + xy$ HAS A PERFECT MATCHING

$G + zt$ HAS A PERFECT MATCHING

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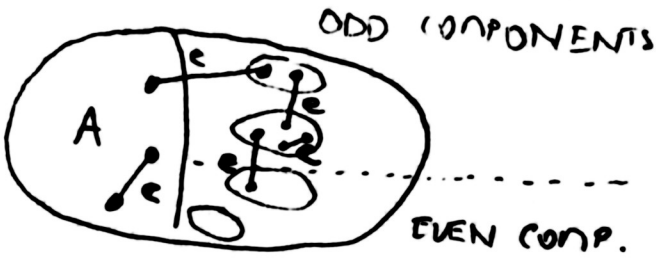
$xy \in M_1$, $zt \in M_2$ (OTHERWISE DONE)

$M_0 := M_1 \text{ xor } M_2 \rightsquigarrow$ PERFECT MATCHING M.

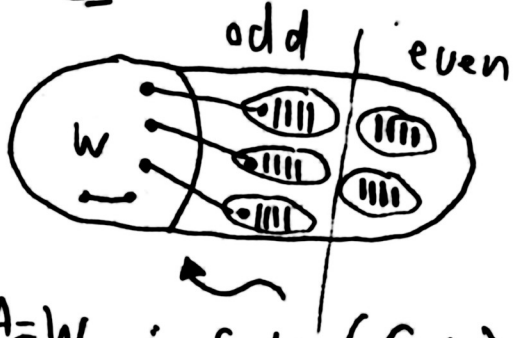
PROOF OF THE ASSUMPTION $G(TC) \Rightarrow G+e\,(TC)$



$c_{odd}(G \setminus A) \le |A|$

FEW CASES

5a)



FOR $A = W$: $c_{odd}(G \setminus W) \le |W|$

LEMMA: H IS A DUCS $\Leftrightarrow$ $K_{1,2} \notin H$ (IS NOT INDUCED SUBGRAPH)

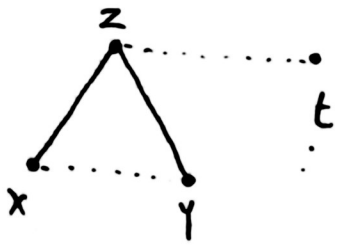
$\Rightarrow$) ✓ $\Leftarrow$) IF H IS NOT DUCS $\Rightarrow$ WE CAN FIND $K_{1,2}$

SHORTEST PATH

HAS 2 EDGES !

5b) $\exists\, xzy$ IN $G \setminus W$

$z \notin W \Rightarrow \exists t : zt \notin E_G \quad t \notin W$



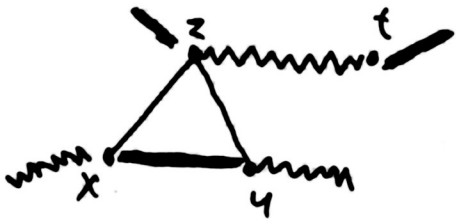
$xy \in M_1 \quad (G + xy), \quad zt \in M_2 \quad (G + zt)$

$M_0 := M_1 \text{ XOR } M_2$

FOR EVERY VERTEX

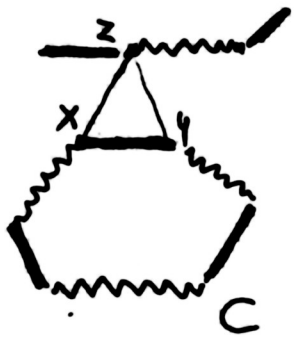
M_0 HAS VERTICES WITH DEGREES 2 OR 0.

$\Rightarrow$ IT IS A DISJOINT UNION OF CYCLES



x, y, z, t BELONG TO CYCLES IN (V_G, M_0)

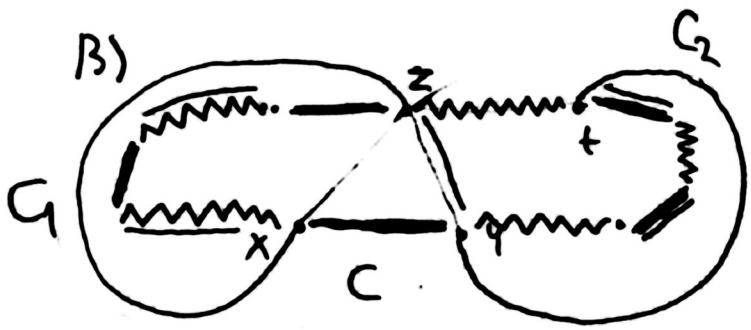
5b) α) x, y AND z, t BELONG TO DIFFERENT CYCLES



$M := (M_1 \setminus C) \cup (M_2 \cap C)$

IS A PERFECT MATCHING

β)



$x, y, z, t \in C$

$M := (M_1 \setminus C) \cup (M_1 \cap G_2) \cup (M_2 \cap G_1) \cup \{zy\}$

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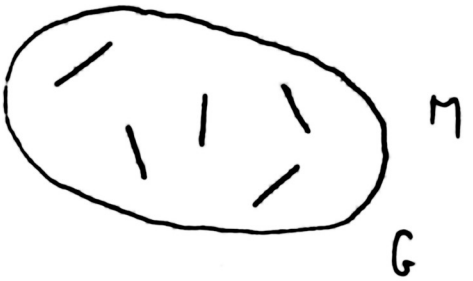
EDMONDS ALGORITHM FOR MAXIMUM MATCHING

DEF.:



A VERTEX u IS FREE IF $u \notin e$ $\forall e \in M$.

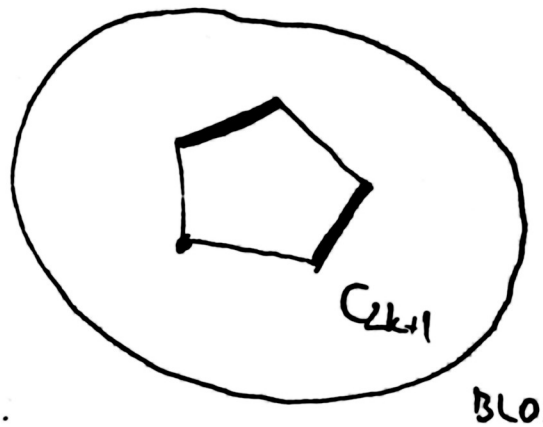
A PATH IS ALTERNATING

ALTER. PATH IS AUGMENTING
IF IT STARTS AND ENDS IN FREE VERTICES

LEMMA: M IS MAXIMUM MATCHING $\Leftrightarrow$ THERE IS NO AUGMENTING PATH.

$\Rightarrow$) SIMPLE

$\Leftarrow$)



BLOSSOM

DEF. C_{2k+1} IS A BLOSSOM IF $|M \cap E(C_{2k+1})| = k$
AND THE LAST VERTEX IS FREE.

LEMMA: IF C IS A BLOSSOM IN (G, M)
$\Rightarrow$ (M IS MAX. MATCHING IN $G \Leftrightarrow$
$M \setminus E(C)$ IS MAX. MATCHING IN $G \cdot C$)

ALG:

INPUT : GRAPH G, MATCHING M

OUTPUT : YES IF M IS MAX. MATCHING

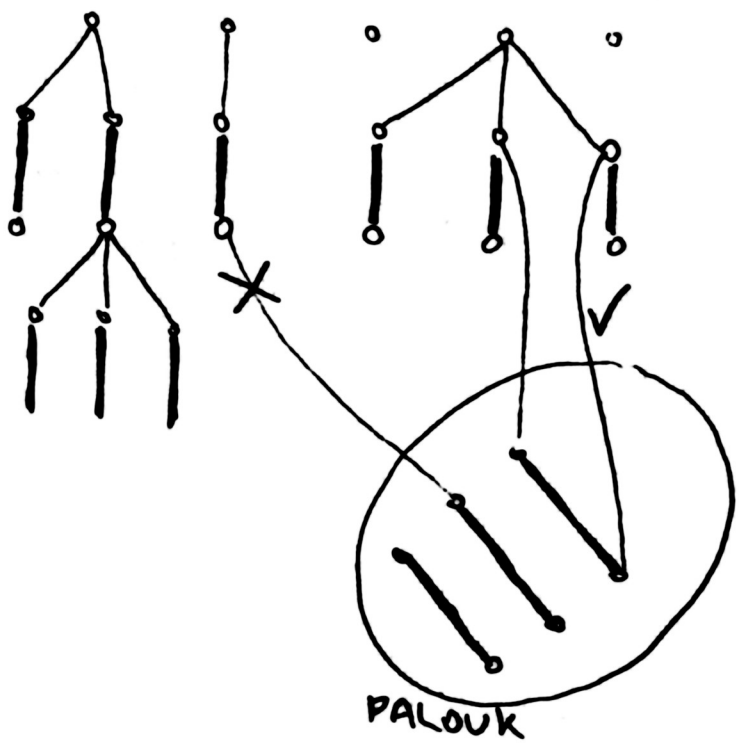
M' MATCHING IN G $|M'| = |M| + 1$

1. BUILD EDMONDS FOREST

LEVEL 0

LEVEL 1

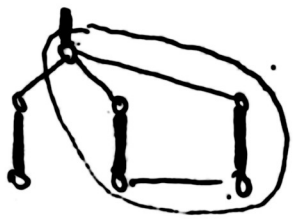
⋮



2. IF ∃ even - even EDGE IN THE FOREST CONNECTING DIFFERENT TREES ⇒ AUGUMENTING PATH

EXTEND M TO M', RETURN M'

IF ∃ even - even EDGE IN THE SAME TREE



⇒ LOCATE AN ALTERNATING CYCLE

SWAP MATCHING EDGES

→ WE GET A GOOD BLOSSOM

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THEN WE CALL $PM(G \cdot C, M \setminus C)$

RETURN
- YES $M \setminus C$ IS MAXIMUM MATCHING
- NO M'

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kam.mff.cuni.cz/~jelinek/cag2.html

LEMMA 3: EDMONDS ALGORITHM IS POLYNOMIAL

LEMMA 1: IF WE DO NOT HAVE MAXIMUM MATCHING, THEN THERE IS AN AUGMENTING PATH

LEMMA 2: IF C IS A BLOSSOM IN (G, M)
$\Rightarrow$ (M IS A MAX. MATCHING IN G $\Leftrightarrow$ $M \setminus E(C)$ IS MAX. MATCHING IN G)

$\Rightarrow$)

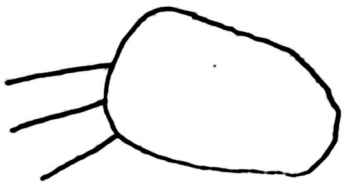
1) G IS 3-REGULAR, 2-EDGE-CONNECTED
$\Rightarrow$ G HAS PERFECT MATCHING.

2) G IS 3-REGULAR $\Rightarrow$ EVERY PERFECT MATCHING CONTAINS ALL THE BRIDGES OF G

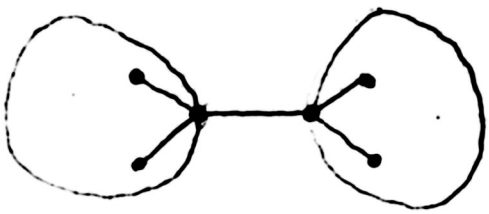
1) TO PROVE $c_{odd}(G \setminus A) \leq |A|$ G IS CONNECTED
FOR CONTRADICTION

AT MOST $3|A|$ EDGES BETWEEN
A AND $G \setminus A$

FROM ODD COMPONENT
AT LEAST <u>3 EDGES</u>



2)



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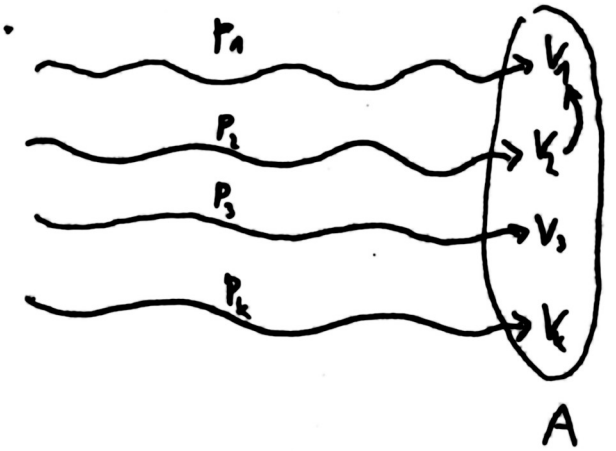
LET G BE A DIRECTED GRAPH. A PATH COVER IS A COLLECTION OF DISJOINT DIRECTED PATHS SUCH THAT EVERY VERTEX IS IN ONE PATH.
THE SIZE OF THE PATH COVER IS THE NUMBER OF PATHS COMPRISING IT.

THEOREM (GALLAI-MILLGRAM): LET G BE A DIRECTED GRAPH. THEN G HAS A PATH COVER OF SIZE AT MOST $\alpha(G)$.

$\alpha(G)$ = INDEPENDENCE NUMBER =
MAXIMUM SIZE OF INDEPENDENT SET =
SET OF VERTICES THAT ARE MUTUALLY NON-ADJACENT

PROOF. WE PROVE A STRONGER STATEMENT:
LET G BE A DIRECTED GRAPH AND $P_1, \dots, P_k$ BE PATHS FORMING ~~A~~ ITS PATH COVER.
IF $k > \alpha(G)$, THERE EXISTS A PATH COVER COMPRISED OF $P_1', \dots, P_{k'}'$ SUCH THAT $k' < k$ AND $\{ter(P_1), \dots, ter(P_k)\} \supseteq \{ter(P_1'), \dots, ter(P_{k'}')\}$
$ter(P)$ IS THE TERMINAL (LAST) VERTEX OF A PATH P.

PROOF (BY INDUCTION ON THE ORDER OF G)
CLEARLY TRUE FOR $|V_G| = 1$ OR $|V_G| = 2$



SINCE $k > \alpha(G)$, A IS NOT AN INDEPENDENT SET

WLOG THERE IS AN ARC $\overleftarrow{v_1 v_2}$

IF P_1 IS A SINGLE-VERTEX PATH, REPLACE P_1 AND P_2 WITH P_2 PROLONGED WITH $\overrightarrow{v_2 v_1}$.

$\Rightarrow$ PATH COVER OF SIZE $k-1$.

LET $\bar{v}_1$ BE THE PREDECESSOR OF v_1 ON P_1.

CONSIDER $G \setminus v_1$ AND THE PATH COVER $P_1 \setminus v_1, P_2, \dots, P_k$.

APPLYING INDUCTION

$\Rightarrow \alpha(G \setminus v_1) \leq \alpha(G)$

$\Rightarrow$ THERE EXISTS A PATH COVER $\bar{P}_1, \dots, \bar{P}_{\bar{k}}$ SUCH THAT $\{ter(\bar{P}_1), \dots, ter(\bar{P}_{\bar{k}})\} \subseteq \{\bar{v}_1, v_2, \dots, v_k\}$ AND $\bar{k} < k$.

(i) THERE IS A PATH $\bar{P}_i$ WITH $ter\, \bar{P}_i = \bar{v}_1$, CONSIDER $\bar{P}_1, \dots, \bar{P}_{\bar{k}}$ WITH $\bar{P}_i$ REPLACED WITH $\bar{P}_i \cup \overrightarrow{\bar{v}_1 v_1}$.

(ii) IF (i) DOES NOT APPLY AND THERE IS A PATH $\bar{P}_i$ ENDING IN v_2, CONSIDER REPLACING $\bar{P}_i$ WITH $\bar{P}_i \cup \overrightarrow{v_2 v_1}$.

(iii) IF NEITHER (i) NOR (ii) APPLY, THEN $\bar{k} \leq k-2$

CONSIDER $\bar{P}_1, \dots, \bar{P}_{\bar{k}}$ AND ADD v_1 AS A SINGLE-VERTEX PATH.

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PARTIALLY ORDERED SET.

CHAIN ... A SET OF VERTICES SUCH THAT EVERY TWO ARE COMPARABLE

ANTICHAIN ... NO TWO ARE COMPARABLE

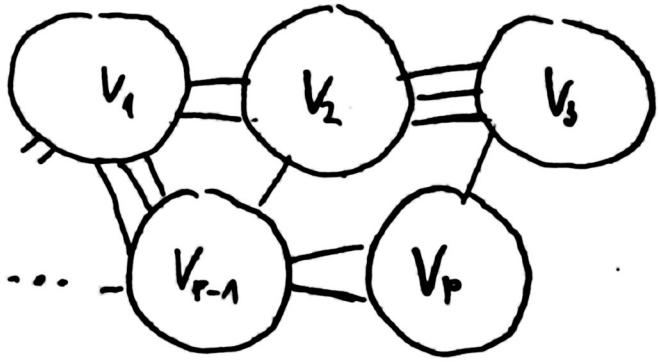
THEOREM (DILLWORTH): LET X BE FINITE PARTIALLY ORDERED SET. THE SIZE OF THE MINIMUM SET OF CHAINS COVERING X IS EQUAL WITH A SIZE OF THE MAXIMUM ANTICHAIN.

PROOF: $\geq$) IF THE ELEMENTS $x_1, \dots, x_r$ FORM AN ANTICHAIN, THEN AT LEAST r CHAINS ARE NEEDED TO COVER X, SINCE NO PAIR x_i, x_j CAN BE CONTAINED IN A COMMON CHAIN.

$\leq$) FORM G ON X WITH AN ARC $x \to y$ IFF $x \leq y$.
$\alpha(G)$ = SIZE OF THE MAXIMUM ANTICHAIN.
BY THE THEOREM OF GALLAI-MILLGRAM THERE IS A COLLECTION OF $\leq \alpha(G)$ COVERING PATHS.

O N C A S THEOREM
OBVIOUS NECESSARY CONDITIONS ARE SUFFICIENT

FOR EVERY PARTITION OF V_G INTO $\overbrace{}^{p}$ DISJOINT SETS.
IF THERE ARE k EDGE-DISJOINT SPANNING TREES
I HAVE TO SEE $\geq k.(p-1)$ EDGES BETWEEN DISJOINT PARTS OF MY PARTITION.



THEOREM (NASH-WILLIAMS)

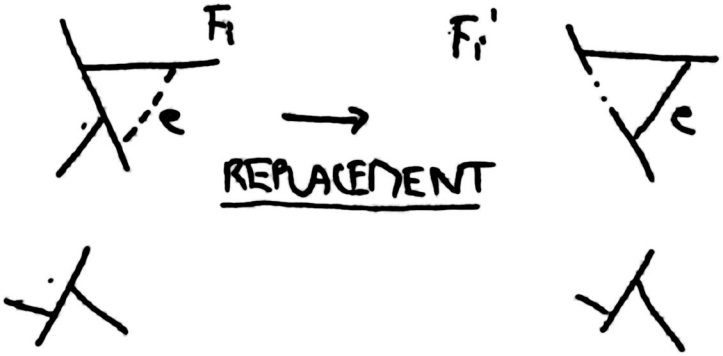
LET G BE A MULTIGRAPH SUCH THAT FOR EVERY $V_G = V_1 \dot\cup \dots \dot\cup V_p$ THE MULTIGRAPH $G[V_1, \dots, V_p]$ HAS AT LEAST $k(p-1)$ EDGES. THEN G HAS k DISJOINT SPANNING TREES.

PROOF. $(F_1, \dots, F_k)$: DISJ. SPANNING FORESTS WITH MAXIMUM # OF EDGES.

(INDUCTION ON THE NUMBER OF VERTICES OF G)

IF THERE ARE $k(n-1)$, THEN F_i's ARE TREES.

OTHERWISE AT LEAST ONE F_i IS DISCONNECTED.



$e \notin \bigcup_{i=1}^{k} F_i$

CONSIDER ALL k-TUPLES OF FORESTS THAT CAN BE OBTAINED FROM $(F_1, \dots, F_k)$ BY A SEQUENCE OF EDGE REPLACEMENTS. LET U BE THE SET OF EDGES THAT CAN ACT AS EDGE REPLACEMENTS.

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$$U \supseteq E_G \setminus \bigcup_{i=1}^{k} F_i \Rightarrow U \neq \emptyset$$

LET G_0 BE THE SUBGRAPH OF G WITH EDGE-SET U.
LET C_0 BE A COMPONENT OF G_0. THEN
$F_i[V(C_0)]$ IS <u>CONNECTED</u>. ($C_0 \geq 2$ VERTICES !!)

$\Rightarrow$ APPLY INDUCTION TO A GRAPH G' OBTAINED FROM G BY CONTRACTING C_0 TO w.

APPARENTLY G' SATISFIES THE ASSUMPTIONS
$\Rightarrow$ WE GET k EDGE-DISJOINT SPANNING TREES
REPLACE w WITH $F_i[V(C_0)]$

LEMMA: $(F_1, \dots, F_k)$ IS A COLLECTION OF FORESTS THAT CAN BE OBTAINED FROM $(F_1', \dots, F_k')$ THROUGH EDGE-REPLACEMENT AND x AND y BE ENDS OF A PATH IN $F_i[V(C_0)]$, THEN x AND y ARE JOINED BY A PATH IN $F_i'[V(C_0)]$ TOO.

1. $r, s \in \mathbb{N}$. $\forall X$ PARTIALLY ORDERED SET WITH $> r.s$ ELEMENTS $\Rightarrow$ X HAS A CHAIN OF SIZE $> r$ OR X HAS AN ANTICHAIN WITH $> s$ ELEMENTS.

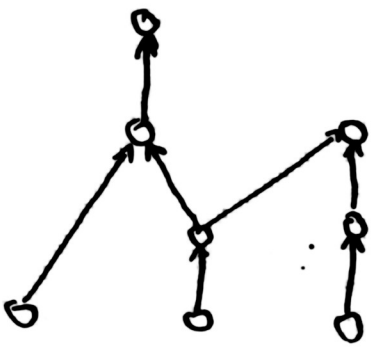
LET US ASSUME THAT EVERY CHAIN HAS AT MOST r ELEMENTS AND EVERY ANTICHAIN HAS AT MOST s ELEMENTS.

LET A BE MAXIMAL ANTICHAIN WITH $k \leq s$ ELEM. LET US CONSIDER MINIMAL COVERING CHAINS $P_1, \dots, P_k$. THEN THE NUMBER OF VERTICES IS $|P_1| + \dots + |P_k| \leq k.r \leq r.s < |X|$ ↯.

2. X POSET CAN BE COVERED BY $\leq k$ ANTICHAINS $\Leftrightarrow$ EACH CHAIN HAS SIZE $\leq k$.

$\Rightarrow$) LET $A_1, \dots, A_r$ BE ANTICHAINS COVERING X, $r \leq k$. LET P BE A CHAIN, THEN EACH VERTEX IS IN EXACTLY ONE A_i AND NO TWO VERTICES OF P CAN BE IN THE SAME A_i

$\Leftarrow$) FROM HASS DIAGRAM



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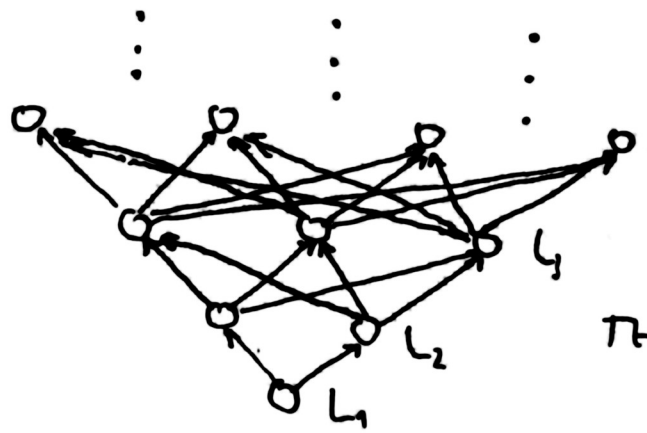
3. FIND POSSET X: X HAS NO INFINITE ANTICHAIN, BUT X CANNOT BE COVERED BY FINITELY MANY CHAINS.

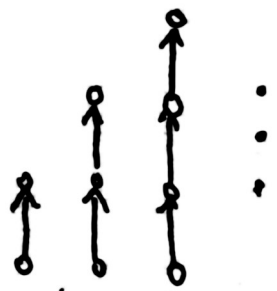


LET $A \subseteq X$ BE ANTI-CHAIN

$\Rightarrow A \subseteq L_n \Rightarrow |A| \leq n$

IF $P_1, \dots, P_k$ COVER X THEN LET US CONSIDER L_{k+1}

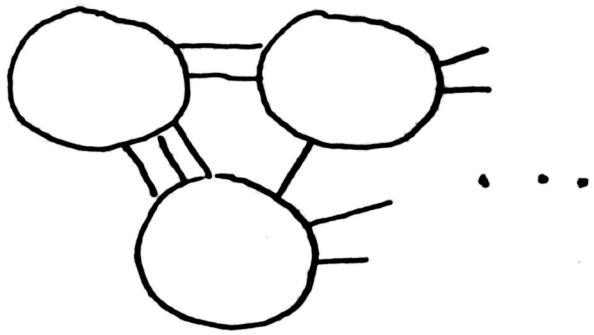
POSET X: NO INFINITE CHAIN, CANNOT BE COVERED BY FINITELY MANY ANTICHAINS



$k \in \mathbb{N}$: G IS $2k$-EDGE CONNECTED

$\Rightarrow$ G HAS k DISJOINT SPANNING TREES.

$k > 1$ $\exists$ $(2k-1)$-EDGE-CONNECTED G WITHOUT k D.S.T.



p COMPONENTS

HAS AT LEAST $k.p$ EDGES, IE $m \geq k.p$

IF IT HAS $< k.p$ EDGES $\Rightarrow$

$$\sum_{i=1}^{p} \deg C_i = 2m < 2kp$$

$\Rightarrow$ $\exists$ C_i WITH LESS THAN $2k$ EDGES.

$$m \geq \frac{(2k-1)p}{2}$$

$$\sum \deg C_i = 2m < (2k-1)p$$

$$\frac{(2k-1)p}{2} \leq m < k(p-1)$$

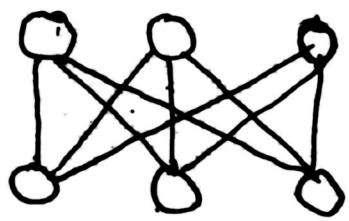
$$\left(k - \frac{1}{2}\right)p \leq m < k(p-1)$$

$$kp - \frac{1}{2}p \leq kp - k$$

$$k < \frac{1}{2}p \qquad p > 2k$$

$k = 1$, $p = 6$ $\qquad m = 9$

$$\underbrace{9}_{\frac{(2k-1)p}{2}} \leq \underbrace{9}_{m} < \underbrace{10}_{k(p-1)}$$



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DM1012 KOM. A GRAFY. II

SATISFIABILITY

$$\phi = \bigwedge_{i=1}^{n} C_i \qquad C_i = \bigvee_{j=1}^{r_i} l_{ij} \qquad l_{ij} = \begin{cases} x \\ \neg x \end{cases} \quad x \in X$$

ϕ IS SATISFIABLE $\Leftrightarrow \exists \varphi : X \to \{T, F\}$
$\forall i \in \{1, \dots, n\} \ \exists j \in \{1, \dots, r_i\} \ \varphi(l_{ij}) = T$.

SAT INPUT: ϕ
QUESTION: Is ϕ SATISFIABLE?
WELL KNOWN: NP-COMPLETE.

3-SAT IS ALSO NP-COMPLETE. 3-SAT IS SAT RESTRICTED TO 3-FORMULAS ($r_i = 3 \ \forall i \in \{1, \dots, n\}$)
$l_{ij} \neq l_{ik} \ \forall j \neq k$

$(x \vee y \vee \neg z) \wedge (\neg x \vee \neg u \vee t) \wedge \dots$
$(x \vee \neg x \vee y)$ IS ALWAYS SATISFIED.

DEF.: k-FORMULA - $\forall$ CLAUSE HAS k (DIFFERENT) LITERALS.
k-SAT ... SAT RESTRICTED TO k-FLAS.

THEOREM: k-SAT IS P FOR $k \leq 2$
NP-COMPLETE FOR $k \geq 3$.

PROOF. EXERCISE.

DEF. $(k, \leq s)$-FORMULA = k-FORMULA AND EVERY VARIABLE OCCURS IN $\leq s$ CLAUSES.

$(k, \leq s)$-SAT .. SATISFIABILITY RESTRICTED TO $(k, \leq s)$-FORMULAE

THEOREM: $k \geq 3$ $\exists f(k)$ SUCH THAT

$(k, \leq s)$-SAT IS
- POLYNOMIAL $s < f(k)$
- NP-COMPLETE $s \geq f(k)$.

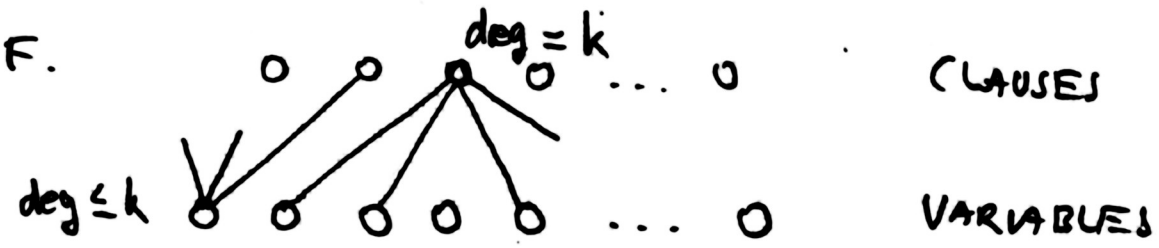
EXAMPLE: FOR $k=3$ $f(k)=4$

$(3, \leq 3)$-SAT IS POLYNOMIAL

$(3, \leq 4)$-SAT IS NP-COMPLETE.

THEOREM: $f(k) \geq k+1$

PROOF.



WE WANT TO SHOW THAT $(k, \leq k)$-SAT IS POLYNOMIAL. TRIVIAL

BECAUSE EVERY $(k, \leq k)$-SAT IS SATISFIABLE.

HALL THEOREM $\Rightarrow$ $\exists$ SYSTEM OF DISTINCT REPRESENTATIVES

$\exists\, g$: CLAUSES $\rightarrow$ VARIABLES SUCH THAT

$g(C_i) \in C_i$ (POSITIVE OR NEGATIVE)

$g(C_i) \neq g(C_j)$ FOR $i \neq j$.

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DM1012 KOMB. A GRAFY II

THEOREM: $\frac{2^{k-2}}{k} \leq f(k) \leq 2^k$

ALTERNATIVE DEF.: $f(k) = \min\{s \mid \exists (k, \leq s)\text{-FLA}$ THAT IS NOT SATISFIABLE $\}$

PROOF: IF THERE EXIST A NONSATISFIABLE FORMULA ψ $(k, \leq s)$-FORMULA

$\Rightarrow$ $(k, \leq s)$-SAT IS NP-COMPLETE.

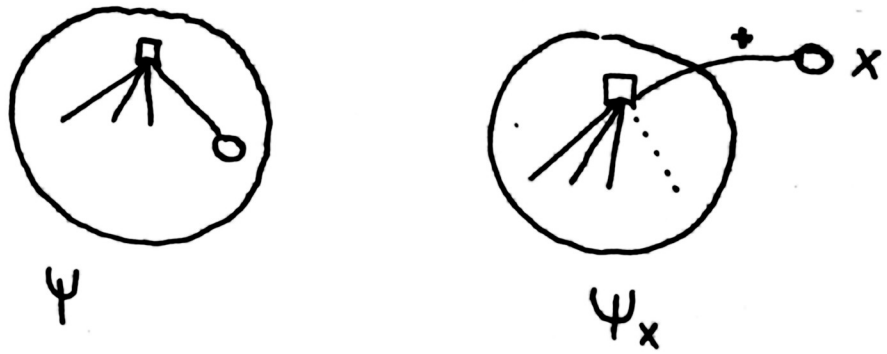
LET ψ BE A <u>MINIMAL</u> NON-SAT $(k, \leq s)$-FLA.

$\square\ \square\ \square\ \dots\ \square$ CLAUSES

$\circ\ \circ\ \circ\ \dots\ \circ$ VARIABLES

IF YOU DELETE ANY CLAUSE, IT WILL BECOME A SATISFIABLE FORMULA

ψ IS NONSAT. AND $\forall C_i \in \psi$: $\psi \setminus C_i$ IS SATISFIABLE.

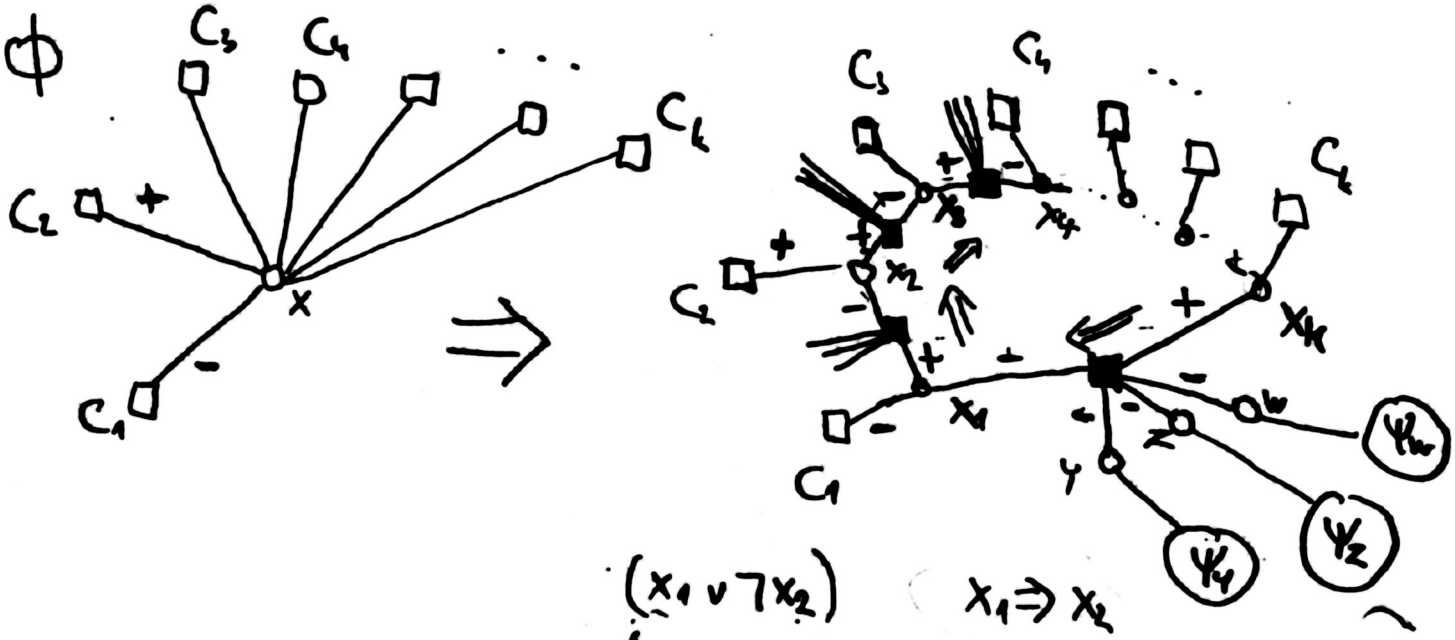


ψ_x IS SATISFIABLE, x IS TRUE IN ALL SATISFYING ASSIGNMENTS. (EXERCISE)

LET ϕ BE $(k, \leq \infty)$-FLA
(k-SAT IS NP-COMPLETE)

$\phi \xrightarrow{\text{WE CONSTRUCT}} \phi'$ $(k, \leq s)$-FORMULA

ϕ IS SAT. $\Leftrightarrow$ ϕ' IS SAT.



$(x_1 \vee \neg x_2)$ $\quad x_1 \Rightarrow x_2$

$(x_2 \vee \neg x_3)$ $\quad x_2 \Rightarrow x_3$

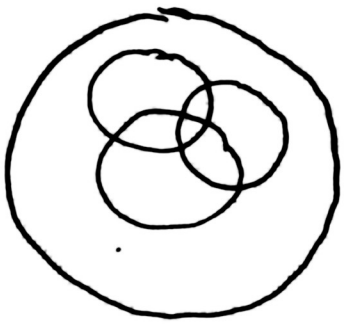
$\vdots$ $\quad \vdots$

$(x_{k-1} \vee \neg x_k)$ $\quad x_{k-1} \Rightarrow x_k$

$(x_k \vee \neg x_1)$ $\quad x_k \Rightarrow x_1$

$f(k) \leq 2^k$ (EXERCISE)

LOVÁSZ LOCAL LEMMA



$A_1, A_2, \ldots, A_n$ EVENTS

$P(A_i) = p_i < 1$

$P(\bigwedge \overline{A_i}) > 0$?

IF A_i ARE MUTUALLY INDEPENDENT

$P(\bigcap \overline{A_i}) = \prod P(\overline{A_i}) = \prod (1 - p_i) > 0$

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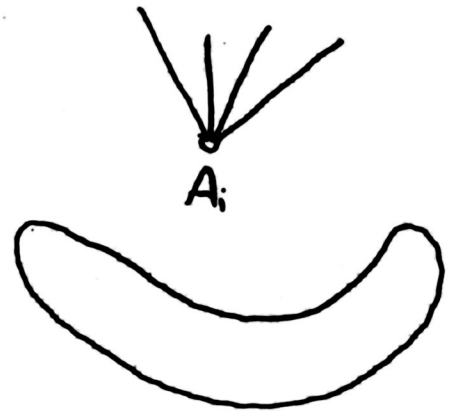
DM1072 KOMB. A GRAFY II

DEF. A DEPENDENCY GRAPH OF $\{A_i\}_{i=1}^n$ IS A GRAPH $G = (\{A_i\}_{i=1}^n, E(G))$



$\{A_i\}_{i=1}^n \setminus N_G[A_i]$

$\forall i$ A_i IS COMPLETELY INDEPENDENT ON $\{A_i\}_{i=1}^n \setminus N_G[A_i]$

A, B ARE INDEPENDENT ... $P(A \cap B) = P(A) \cdot P(B)$

A_i A_3, A_4, A_6, A_9 ... A_i INDEP. $A_3 \cap A_4 \cap \overline{A_6} \cap A_9$

A	EVEN		$1/2$	$= \frac{3 \cdot 6}{6 \cdot 6}$
B		EVEN	$1/2$	
C	SUM IS EVEN		$1/2$	$= \frac{6 \cdot 3}{6 \cdot 6}$

$P(A \cap B) = \frac{1}{4}$, $P(A \cap C) = \frac{3 \cdot 3}{6 \cdot 6} = \frac{1}{4}$, $P(B \cap C) = \frac{1}{4}$

$P(A \cap B \cap C) = \frac{1}{4}$!!!

$(A \cap B)$ AND C ARE <u>NOT</u> INDEPENDENT.

G IS DEPENDENCY GRAPH OF THE EVENTS A_i

MAX DEGREE OF G IS d

$P(A_i) \leq p$

$pd \leq \frac{1}{4}$ $\Rightarrow$ $P(\wedge \bar{A}_i) > 0$

PROOF OF $f(k) > \frac{2^{k-2}}{k}$

$(k, \leq s)$ - FORMULA ϕ $\Rightarrow$ ϕ IS SATISFIABLE

RANDOM ASSIGNMENT TO THE VARIABLES

$P(\varphi(x) = T) = \frac{1}{2}$ AND INDEPENDENT

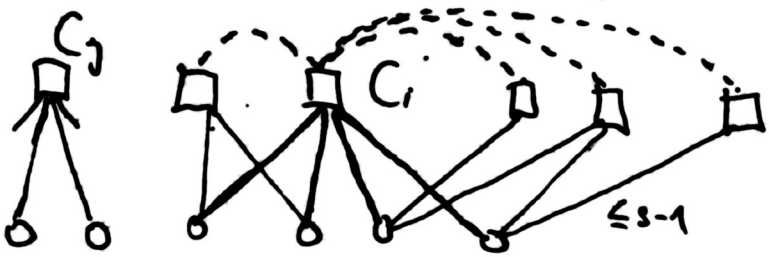
C_i CLAUSES OF ϕ

"BAD EVENTS" $A_i = \{\varphi(C_i) = F\}$

$|C_i| = k$ ... $\left(\frac{1}{2}\right)^k$ PROBABILITY OF : NONE OF THE LITERALS IS SATISFIED IN THIS C_i.

$P(A_i) = \frac{1}{2^k}$

DEPENDENCY GRAPH



EVERY C_i IS TOTALLY INDEPENDENT ON $\{C_j \mid var(C_i) \cap var(C_j) = \emptyset\}$

APPARENTLY $d \leq k(s-1)$

$pd \leq \frac{1}{2^k} \cdot k(s-1) < \frac{1}{2^k} \cdot k \frac{2^{k-2}}{k} = \frac{1}{4}$

$\Rightarrow P(\wedge \bar{A}_i) > 0$ $\left[\text{BEST KNOWN } f(k) \leq c \frac{2^k \lg k}{k}\right]$

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DM101L KOMB. A GRAFY II

1-IN-3-SAT ϕ - 3-FORMULAS

SATISFYING A CLAUSE = $\forall$ CLAUSES
EXACTLY ONE LITERAL IS TRUE

1-IN-k-SAT IS NP-COMPLETE EVEN IF
- ALL LITERALS ARE POSITIVE
- EVERY VARIABLE OCCURS EXACTLY IN k CLAUSES

NAE-SAT $\forall$ CLAUSE ≥ 1 LIT. TRUE ≥ 1 LIT. FALSE
NP-COMPLETE t-FORMULA $(t \geq 3)$
NO NEGATIVE

EXERCISE: NAE-SAT

$\forall$ CLAUSE $|C_i| = k$
$\forall$ VARIABLE IN k CLAUSES
k IS LARGE ENOUGH $(k \geq 9)$
$\Rightarrow$ ϕ IS NAE-SATISFIABLE

EVERY k-UNIFORM k-REGULAR MULTIGRAPH IS BICOLOURABLE

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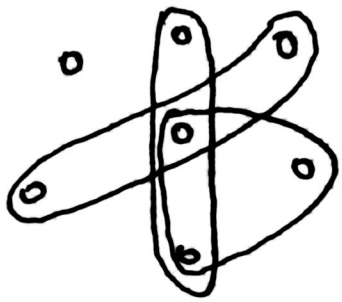
EX: DM1 012 KOMB. A GRAFY II

HYPERGRAPH (V, E) HYPEREDGES $h \subseteq V$

k- UNIFORM HYPERGRAPH $\forall h \in E \ |h| = k$



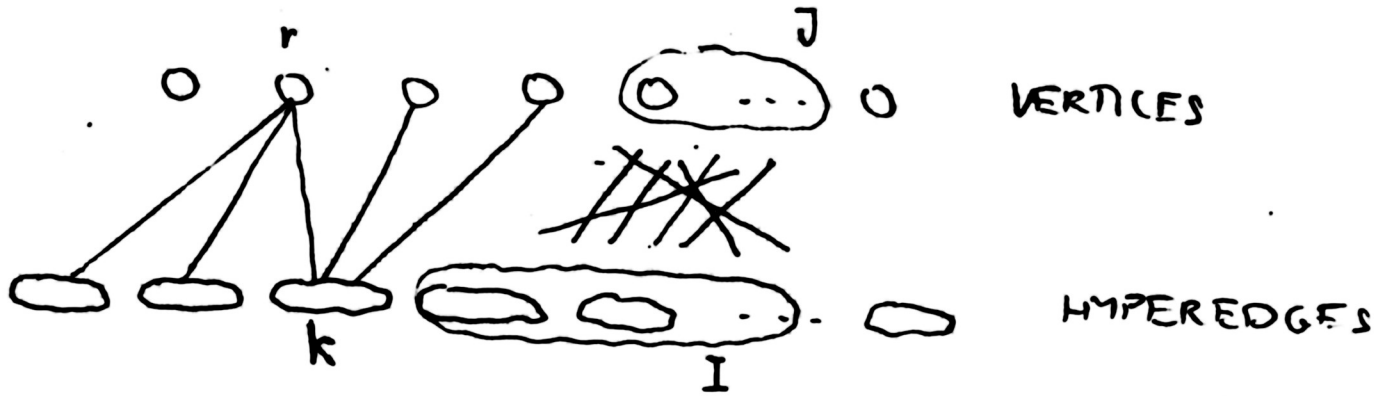
k - REGULAR HYPERGRAPH $\forall v \in V$ v BELONGS TO EXACTLY k HYPEREDGES

BICOLOURING $f: V \to \{red, blue\}$

EVERY HYPEREDGE HAS BOTH COLOURS

k-HYPERGRAPHS = k-UNIFORM HYPERGRAPHS.

1) $\forall k \geq 3 \ \exists r_0 \ \forall r \leq r_0$: EVERY k-UNIFORM r-REGULAR HYPER GRAPH HAS A BICOLOURING.
r_0 IS AS LARGE AS POSSIBLE.



FOR $k \geq 2r$ TWO REPRESENTANTS FOR EVERY HYPEREDGE

$k \cdot |I| = r \cdot |J| \Rightarrow |J| = \frac{k}{r} |I| \geq 2|I|$ $\boxed{r_0(k) \geq \frac{k}{2}}$

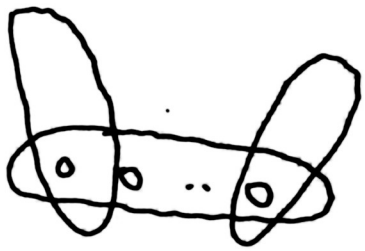
$$r_0(k) \geq \frac{2^{k-3}}{k}$$

PROOF. WE COLOUR THE VERTICES.

BAD EVENTS: $P(A_i) = \frac{2}{2^k} = p$ 2 BAD COLOURINGS

DEPENDENCY GRAPH: VERTICES = HYPER EDGES

EDGES = NONDISJOINT HYPEREDGES



FOR $\boxed{r_0 \geq \frac{2^{k-3}}{k}}$

$$d = k(r_0 - 1) < k r_0 \leq k \frac{2^{k-3}}{k} = 2^{k-3}$$

$$\Rightarrow \quad pd \leq \frac{1}{4}$$

UPPER BOUND ON $r_0(k)$? IF k IS CONSTANT ?
ALGORITHM ?

$$\frac{2^{k-3}}{k} \leq r_0(k) \leq \frac{\binom{2k}{k}}{2k}$$

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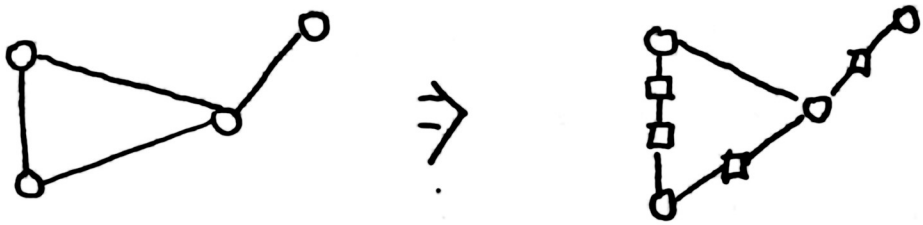
DM1012 Kom. A grafy II

G ... A SUBDIVISION OF G IS ANY GRAPH OBTAINED BY INTRODUCING ONE OR MORE VERTICES OF DEGREE TWO ON EDGES OF G.



THEOREM (MADER): $\forall r \exists D$ S.T. ANY GRAPH WITH AVERAGE DEGREE AT LEAST D CONTAINS A SUBDIVISION OF K_r.

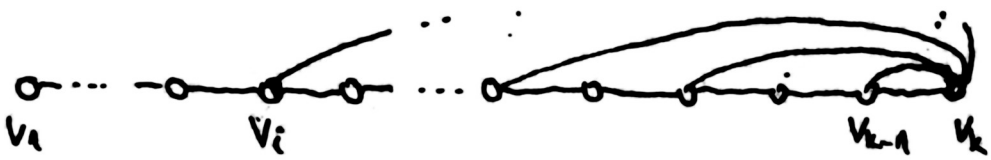
$D = 2^{\binom{r}{2}}$

FOR A FIXED r BY INDUCTION ON $m = r, \dots, \binom{r}{2}$ $r \geq 8$
EVERY GRAPH WITH AVERAGE DEGREE $\geq 2^m$
CONTAINS A SUBDIVISION OF A SIMPLE GRAPH (WITH NO MULTIPLE EDGES)
WITH r VERTICES AND m EDGES.

FOR $m = r$ WE WANT TO FIND A SUBDIVISION OF C_r = ANY CYCLE OF LENGTH $\geq r$.
WE KNOW THAT avg. deg. $\geq 2^r$, #EDGES $\geq 2^{r-1} \cdot n$
WHERE n IS A NUMBER OF VERTICES.

G CONTAINS A SUBGRAPH G' WITH MINIMUM DEGREE $\geq 2^{r-1}$
(KEEP REMOVING VERTICES WITH DEGREE $< 2^{r-1}$)
FOR $r \geq 3$ $2^{r-1} \geq r$, SO mindeg $G' \geq r$.

TAKE THE LONGEST PATH IN G'



v_i .. NEIGHBOUR OF v_k WITH SMALLEST i.

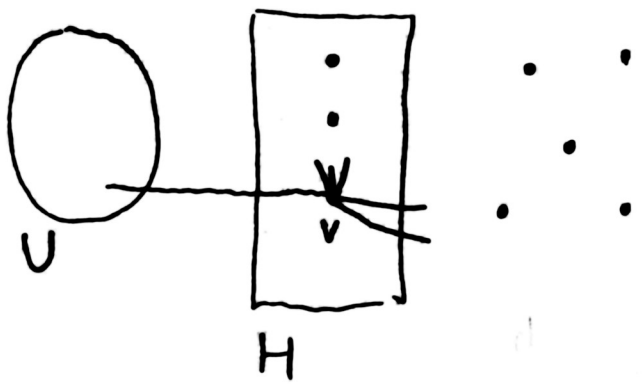
INDUCTION STEP ($m > r$) WLOG G IS CONNECTED

CONSIDER THE LARGEST <u>CONNECTED</u> SUBGRAPH U OF G THAT G/U HAS AT LEAST $2^{m-1} \cdot n'$ EDGES WHERE n' IS THE NUMBER OF VERTICES OF G/U.

$U = \{v\}$, U HAS THE DESIRED PROPERTY (G HAS $\geq 2^{m-1} \cdot n$ EDGES $G/\{v\} = G$)

$H :=$ THE SUBGRAPH OF G INDUCED BY THE NEIGHBOURS OF U NOT CONTAINED IN U.

WE CLAIM THAT $\underline{\min.\deg. H \geq 2^{m-1}}$.



ASSUME THAT $\deg_H v < 2^{m-1}$ FOR SOME $v \in H$.

CONSIDER $U' := U \cup \{v\}$

U' IS A CONNECTED SUBGRAPH OF G.

G/U' HAS $n'-1$ VERTICES

AN EDGE FROM U TO v

THE NUMBER OF EDGES OF G/U' $\geq 2^{m-1} n' - 1 -$
$- \underbrace{(2^{m-1} - 1)}_{\deg_H v \leq 2^{m-1}-1} = 2^{m-1}(n'-1)$

SO U IS NOT THE LARGEST SET ↯ (CONTRADICTION)

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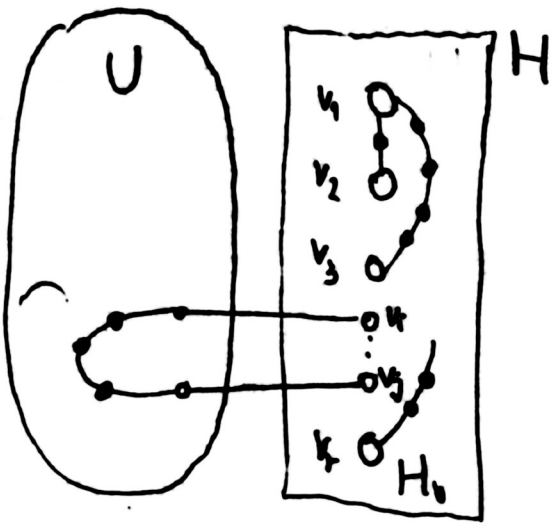
H CONTAINS A SUBDIVISION H_0 OF A GRAPH $\overline{H_0}$ WITH r VERTICES AND $m-1$ EDGES (BY INDUCTION).

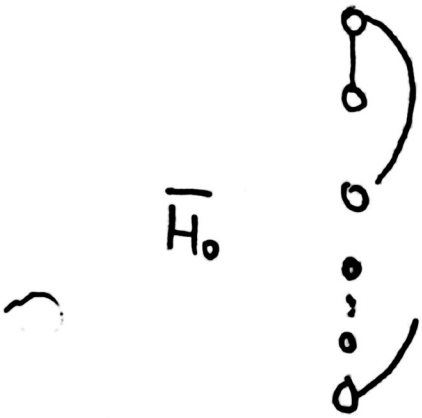


SINCE

$m \leq \binom{r}{2}$

WE CAN CHOOSE A PAIR OF VERTICES v_i, v_j NOT JOINED BY A PATH IN A SUBDIVISION H_0.

NOW WE CAN CONNECT THESE VERTICES v_i, v_j BY A PATH THROUGH U.



ORDER $(\overline{H_0}) = r$

EDGES $(\overline{H_0}) = m-1 \quad m \leq \binom{r}{2}$

■

A GRAPH G WITH AT LEAST $2k$ VERTICES IS k-LINKED IF FOR EVERY VERTICES $s_1, \dots, s_k$ AND $t_1, \dots, t_k$ (DISTINCT) G CONTAINS k-VERTEX DISJOINT PATHS P_i WHERE P_i JOINS s_i AND t_i FOR $i = 1, \dots, k$.

k-LINKED $\Rightarrow$ k-CONNECTED

(BUT k-CONNECTED $\not\Rightarrow$ k-LINKED IF $k \geq 2$)

THEOREM: $\forall k \, \exists K$ SUCH THAT EVERY K-CONNECTED GRAPH IS k-LINKED.

$K = 2k + 2^{\binom{3k}{2}}$

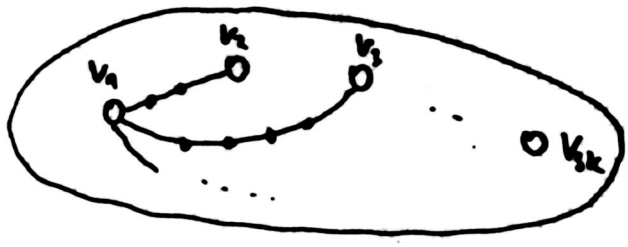
G IS K-CONNECTED,

$s_1, \dots, s_k, t_1, \dots, t_k$ DISTINCT VERTICES

$:= G \setminus \{s_1, \dots, s_k, t_1, \dots, t_k\}$ IS $2^{\binom{3k}{2}}$-CONNECTED

min. deg. $H \geq 2^{\binom{3k}{2}}$

$\Rightarrow$ (MADER THEOREM) WE HAVE A SUBDIVISION H OF K_{3k}.



s_1 o ---o s_k o t_1 ... o t_k

WE HAVE $\binom{3k}{2}$ DISJOINT PATHS BETWEEN $v_1, \dots, v_{3k}$

WE CLAIM THAT $\exists$ VERTEX DISJOINT PATHS FROM $s_1, \dots, s_k, t_1, \dots, t_k$ TO A SUBSET OF $2k$ VERTICES OF $\{v_1, \dots, v_{3k}\}$.

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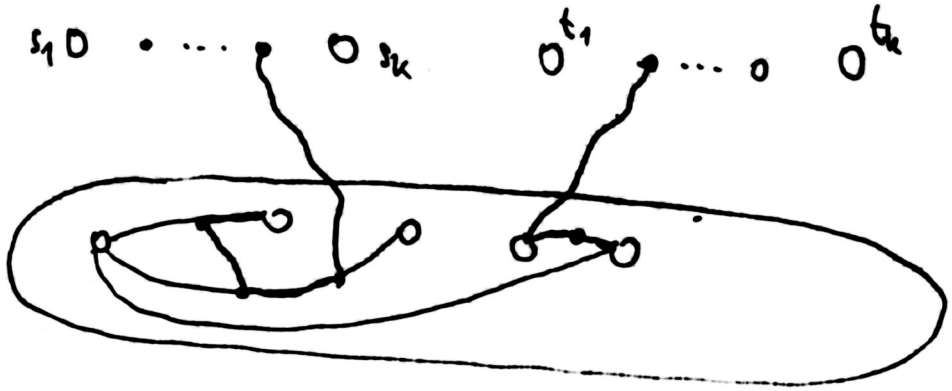
DM1012 KOMBINATORIKA A GRAFY II

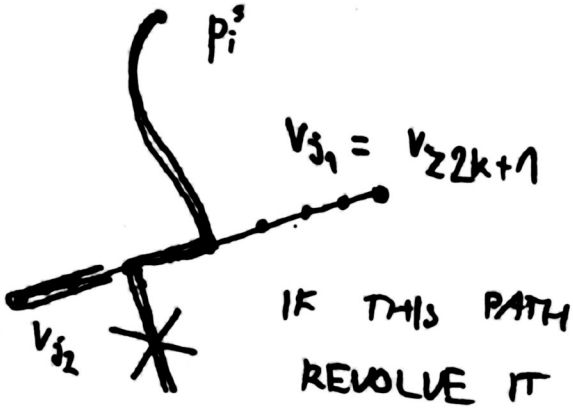
CHOOSE $P_1^s, \dots, P_k^s, P_1^t, \dots, P_k^t$ IN SUCH A WAY THAT THE NUMBER OF EDGES OUTSIDE H CONTAINED IN P_i^s, P_i^t IS MINIMAL (AND SUBJECT TO THESE THE PATHS ARE <u>SHORTEST POSSIBLE</u>)

WLOG P_i^s ENDS AT v_i, P_i^t ENDS AT v_{i+k}

VERTICES $v_{2k+1}, \dots, v_{3k}$ ARE NOT ON ANY OF THESE PATHS.



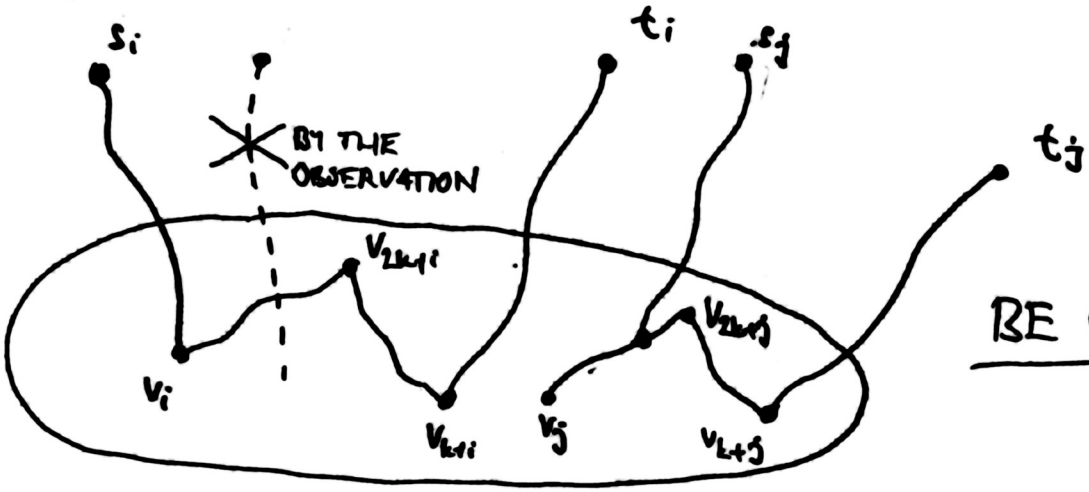
IF A PATH P_i^s CONTAINS A VERTEX FROM v_{j_1} TO v_{j_2} IN H AND $j_1 \geq 2k+1$, THEN $i = j_2$ AND P_i^s DOES NOT LEAVE H AFTERWARDS.



IF THIS PATH LEAVES H AFTERWARDS, REVOLVE IT TO $v_{\geq 2k+1}$!!

THIS SHOULD BE IMPOSSIBLE BY THE CHOICE OF THE PATHS!

$s_i - t_i$ FOLLOWS p_i^s TO v_i, USE THE PATHS IN H CORRESPONDING TO EDGES $v_i - v_{2k+i}$ AND $v_{2k+i} - v_{k+i}$ AND THE PATH p_i^t



BE CAREFUL !

■

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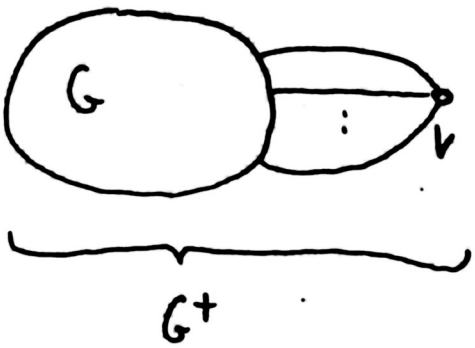
EX: DM1012 KOMB. A GRAFY II

$G=(V,E)$, G IS k-CONNECTED IFF $|V| \geq k+1$

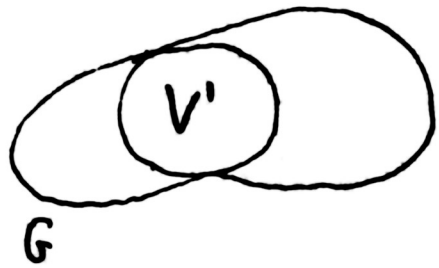
1. AND G HAS NO VERTEX CUT OF SIZE $< k$.
2. FOR EACH TWO VERTICES x,y, $x \neq y$ THERE EXIST TWO INTERNALLY DISJOINT x-y PATHS.
3. $\forall S,T \subseteq V$ $|S|=|T|=k$ $\exists k$ DISJOINT S-T PATHS.

LEMMA: G k-CONNECTED $\Rightarrow$ G^+ IS k-CONNECTED.



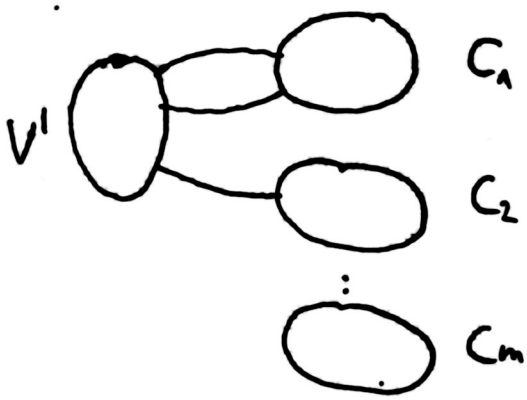
$\deg_{G^+} v = k$

(3) $\Rightarrow$ (1) $|V| \geq k+1 \wedge \forall S,T \subseteq V$ $|S|=|T|=k \Rightarrow \exists k$ S-T PATHS (DOD.)
ASSUME THAT G HAS A CUT OF SIZE $k' < k$.



$|V'| = k' < k$

$G \setminus V'$ IS DISCONNECTED.



IF $|C_1| \geq k$ AND $|C_2| \geq k$ THEN FOR S A k-SUBSET OF C_1 AND T A k-SUBSET OF C_2 WE GET A CONTR. ↯

IF $|C_1| \geq k$ ONLY THEN FOR $S=T=$ k-SUBSET OF C_1 WE GET A CONTR. ↯

IF FOR ALL i: $|C_i| < k$. ???

$\neg$ (1) $\Rightarrow$ $\neg$ (3)

WE CAN ASSUME THAT $|V'| = k-1$.

$S := V' \cup \{s\}$, $T := V' \cup \{t\}$.

BETWEEN S AND T IT IS IMPOSSIBLE TO FIND k VERTEX DISJOINT PATHS. ↯

(2) $\Rightarrow$ (3.)

FIRST WE CHOOSE S, T ARBITRARILY

AND ADD VERTICES s, t.

$G + \{s,t\}$ IS k-CONNECTED

$\Rightarrow$ WE CAN FIND k DISJOINT s-t PATHS

$\Rightarrow$ WE GET k DISJOINT S-T PATHS. ∎

30.10.2007 TU
PETER ČERNO

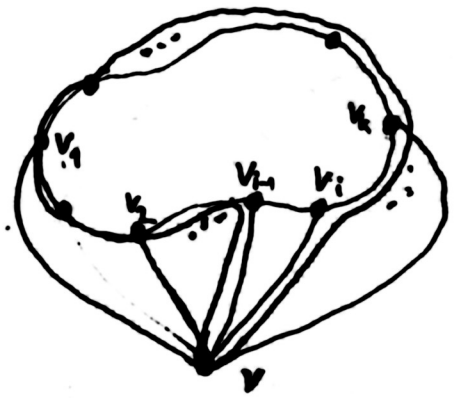
EX: DMIO12 KOMB. A GRAFY II

1) G k-CONNECTED, $|V| \geq 2k$
$\Rightarrow$ G HAS A CYCLE OF LENGTH $\geq 2k$.
$\exists$ "LARGE" k-CONNECTED G WITHOUT A CYCLE LONGER THAN $2k$.

G IS k-CONNECTED $\Rightarrow$ MIN. DEGREE OF G IS k
$\Rightarrow$ THERE EXIST A PATH OF LENGTH k
TAKE THE LONGEST PATH

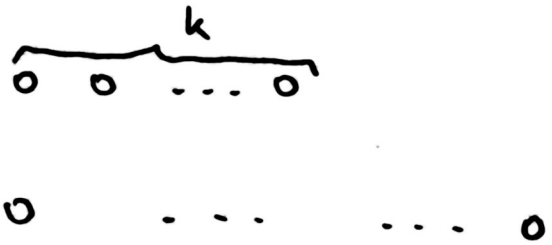
v_r HAS AT LEAST k NEIGHBOURS, ALL FROM $\{v_1, \dots, v_{r-1}\}$
SO $r-1 \geq k$, $r \geq k+1$ $\Rightarrow$ WE HAVE A k-PATH. ???

LET US TAKE THE LONGEST CYCLE IN G. IF $\ell < 2k$



LETS TAKE $v \in G \setminus C$.
WE CAN FIND k VERTEX-DISJ. PATHS FROM v TO THIS CYCLE. (WE TAKE THE SHORTEST PATHS)
AT LEAST 2 ENDPOINTS ARE THE NEIGHBOURS ON THE CYCLE!

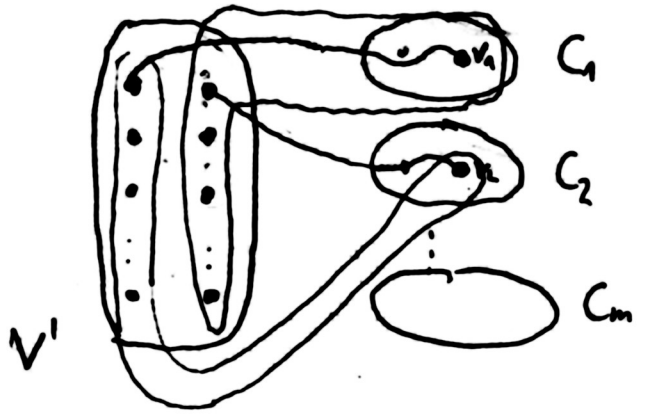
IF WE TAKE $K_{k,N}$ BIG N



EVERY CYCLE HAS TO VISIT THE FIRST PARTITION
$\Rightarrow$ THE LENGTH OF THE CYCLE $\leq 2k$!!

2) $k \geq 2$, G IS k-LINKED $\Rightarrow$ $(2k-1)$-CONNECTED.

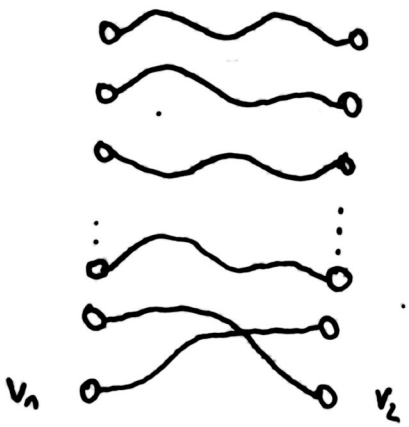
LET US ASSUME THAT THERE IS A SET $V' \subseteq V$ $|V'| = 2k-2$ SUCH THAT $G \setminus V'$ IS DISCONNECTED

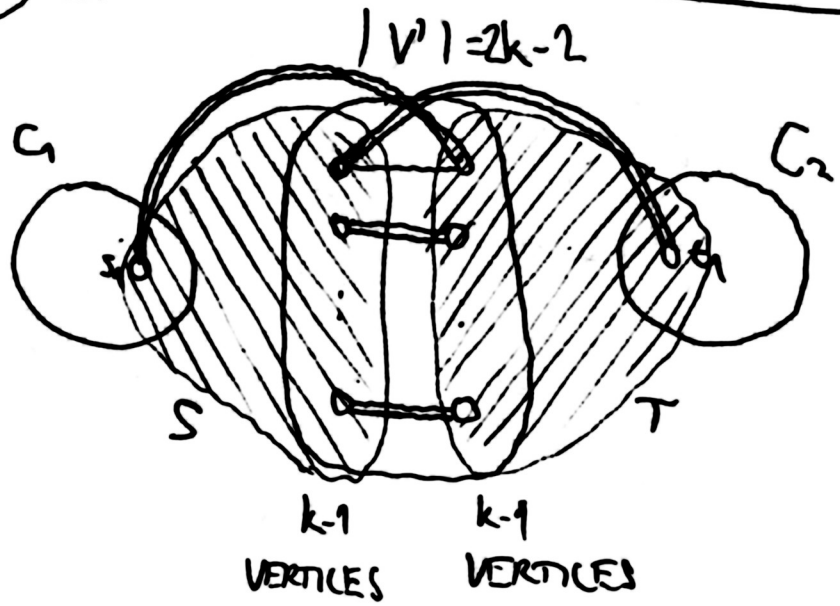


k-LINKED

USPORIADANIE v S A T !

MÔŽME PREDPÍSAŤ ABY v_1 A v_2 BOLI SPOJ





G . k VERTEX DISJOINT $S-T$ PATHS.

06.11.2007 FTN
PETER ČERNO

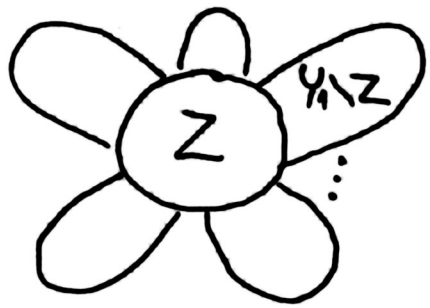
DM1012 KOMBINATORIKA A GRAFY II

SUNFLOWER LEMMA (ERDŐS-RADO)

A SET SYSTEM $Y_1, \dots, Y_k$ IS CALLED A SUNFLOWER IFF $\exists Z$ S.T. $Z = Y_i \cap Y_j$ FOR ANY $i \neq j$

THE CORE OF THE SUNFLOWER

SO $Y_1 \setminus Z, \dots, Y_k \setminus Z$ ARE DISJOINT



$$\boxed{Z = \bigcap_{i=1}^{k} Y_i}$$

LET $\mathcal{F}$ BE A COLLECTION OF s-ELEMENT SETS. IF $|\mathcal{F}| > s!\,(k-1)^s$ SETS, THEN $\mathcal{F}$ CONTAINS A SUNFLOWER $Y_1, \dots, Y_k$.

PROOF.: BY INDUCTION ON s.

$s = 1$. $|\mathcal{F}| > k-1 \Rightarrow |\mathcal{F}| \geq k$ SO $\mathcal{F}$ CONTAINS AT LEAST k 1-ELEMENT SETS $Y_1, \dots, Y_k$. APPARENTLY $Y_i \cap Y_j = \emptyset$ $i \neq j$, SO $Y_1, \dots, Y_k$ IS A SUNFLOWER.

$s > 1$. CONSIDER THE LARGEST SUBFAMILY OF $\mathcal{F}$ OF MUTUALLY DISJOINT SETS $A_1, \dots, A_t$

IF $t \geq k \Rightarrow$ <u>OK</u> $A_1, \dots, A_k$ IS A SUNFLOWER

OTHERWISE LET $X = A_1 \cup \dots \cup A_t$

$|X| \leq (k-1) \cdot s$ BECAUSE $t \leq k-1$.

FOR EVERY [REMAINING] $Y \in \mathcal{F}$: $Y \cap X \neq \emptyset$

SINCE $\mathcal{F}$ HAS MORE THAN $s!\,(k-1)^s$ SETS

AND $|X| \leq (k-1)s \Rightarrow \exists x \in X$ THAT IS CONTAINED IN MORE THAN $\frac{s!(k-1)^s}{s(k-1)}$ SETS OF $\mathcal{F}$.

LET $x_0 \in X$ BE SUCH AN ELEMENT.

THERE ARE $Z_1, \dots, Z_r \in \mathcal{F}$ WHERE $r > \frac{s!(k-1)^s}{s \cdot (k-1)}$ SUCH THAT $x_0 \in Z_i$ FOR $\forall i \in \{1, \dots, r\}$.

$$\mathcal{F}' := \{ Z_1 \setminus \{x_0\}, \dots, Z_r \setminus \{x_0\} \}$$

APPARENTLY $|\mathcal{F}'| > (s-1)!\,(k-1)^{s-1}$

SO IF WE APPLY INDUCTION, WE GET:

$Y_1', \dots, Y_k' \in \mathcal{F}'$ SUNFLOWER

$\Rightarrow \underbrace{Y_1' \cup \{x_0\}}_{Y_1}, \dots, \underbrace{Y_k' \cup \{x_0\}}_{Y_k}$ IS A SUNFLOWER IN $\mathcal{F}$. ■

THEOREM (ERDŐS-KO-RADO): LET $\mathcal{F}$ BE A FAMILY OF k-ELEMENT SETS AND SUBSETS OF AN n-ELEMENT SET, AND $n \geq 2k$.
IF EVERY TWO SETS IN $\mathcal{F}$ HAVE AN NONEMPTY INTERSECTION, THEN $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

FOR INSTANCE $X = \{1, \dots, n\}$ AND CONSIDER ALL k-ELEMENT SUBSETS CONTAINING 1.

PROOF.: ASSUME THAT THE n-ELEMENT SET IS $\{1, \dots, n\}$.

A PERMUTATION π, A SET $Y \in \mathcal{F}$ IS $\boxed{\pi\text{-NICE}}$ NICE IF THERE EXISTS i SUCH THAT

$$Y = \{\pi(i), \pi(i+1), \dots, \pi(i+k-1)\}.$$

06.11.2007 Th
PETER ČERNO

DM1012 KOMBINATORIKA A GRAFY II

i	1	2	3	4	5	6	7	8 ...
$\pi(i)$	3	1	5	2	6	4	3	1

π-NICE

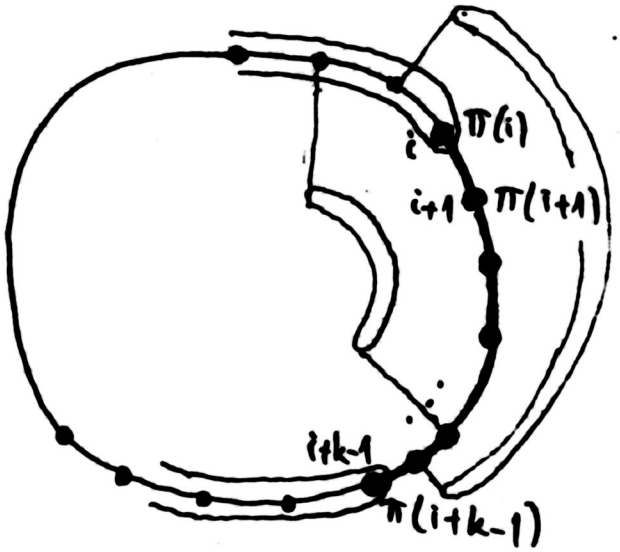
$n = 6$, $k = 3$ $\qquad n \geq 2k$

EVEN FOR $i = 6$ WE HAVE $\{4, 3, 1\}$ π-NICE SET.

THE NUMBER OF SETS CONTAINED IN $\mathcal{F}$ THAT ARE π-NICE IS AT MOST k FOR ANY PERMUTATION π.

PROOF. IF THERE IS NO π-NICE SET $\Rightarrow$ <u>ok</u>.
OTHERWISE FIX ONE π-NICE SET $Y_0 = \{\pi(i), \dots, \pi(i+k-1)\}$.

<u>$n \geq 2k$</u>



Y_0 $\{\pi(2), \pi(3), \pi(4), \pi(5)\}$

$\{\pi(1)\ \pi(1)\ \pi(2)\ \pi(3)\}\{\pi(4), \pi(5), \pi(6), \pi(7)\}$

$\dots \quad \pi(2)\}\ \{\pi(3), \pi(4), \pi(5), \pi(6)\}$ <u>A PAIR</u> (DISJOINT)

AT MOST ONE OF THESE TWO SETS ARE IN $\mathcal{F}$

THERE ARE $k-1$ <u>PAIRS</u> OF π-NICE SETS INTERS. Y_0.

$\Rightarrow$ <u>THERE ARE AT MOST $k-1$ π-NICE SETS IN $\mathcal{F}$. $\neq Y_0$.</u>

LET WE COUNT (π, Y) $Y \in \mathcal{F}$ Y IS π-NICE

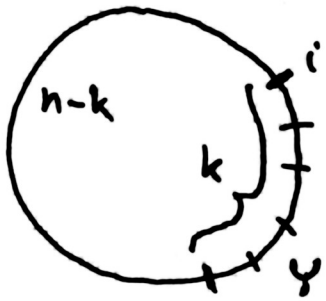
$$|\mathcal{F}| \cdot \underset{\uparrow i}{n} \cdot k!\,(n-k)! \underset{=}{\leq} \#\text{PAIRS} \leq n!\,k$$

WE CHOOSE π

AT MOST k π-NICE SETS Y FOR GIVEN π IN $\mathcal{F}$!!!

WE CHOOSE Y

AND THEN i, AND PERMUTATION π

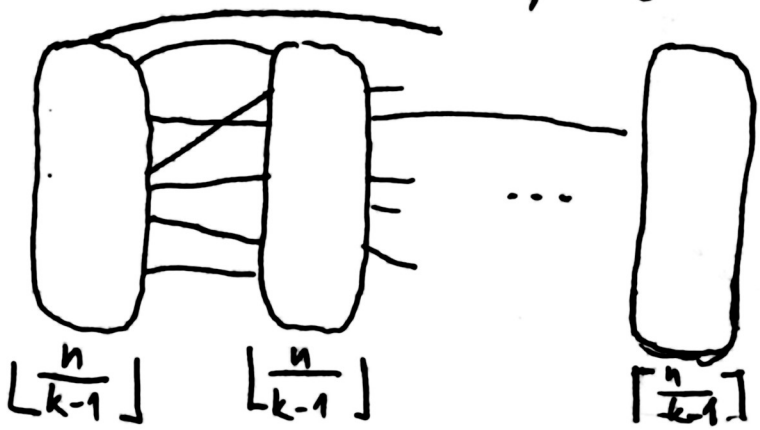


← THESE ARE ELEMENTS OF Y

$$\Rightarrow \quad |\mathcal{F}| \leq \frac{n!\,k}{n\,k!\,(n-k)!} = \binom{n-1}{k-1}. \quad \blacksquare$$

THEOREM (TURAN)

$T(n, v)$ ... n VERTICES, NO CLIQUE OF ORDER k



VERTICES ARE SPLITTED INTO $k-1$ GROUPS

EACH VERTEX FROM ONE GROUP IS CONNECTED WITH EACH VERTEX FROM EACH OTHER GROUP

$n = 178$
$k = 5$

35, 35, 36, 36, 36, 36

APPARENTLY $\omega(T(n,k)) \leq k-1$

THE NUMBER OF EDGES $\|T(n,k)\|$

$$\|T(n,k)\| = \begin{cases} \binom{n}{2} & \text{FOR } n \leq k-1 \\ \underbrace{\|T(n-(k-1),k)\|}_{\text{TOP PART}} + \underbrace{(n-k+1)(k-1)}_{\text{BETW. TOP \& BOTTOM P.}} + \underbrace{\binom{k-1}{2}}_{\text{BOTTOM PART}} \end{cases}$$

06.11.2007 Tu
PETER ČERNO

DM1012 KOMB. A GRAFY II

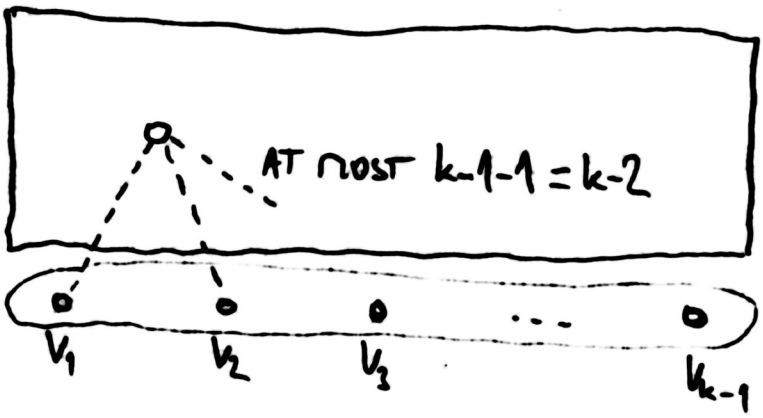
THEOREM (TURAN): LET G BE AN n-VERTEX GRAPH WITH NO K_k AS A SUBGRAPH $\Rightarrow$ THE NUMBER OF EDGES OF G IS AT MOST $\|T(n,k)\|$.

PROOF. BY INDUCTION ON n. (FOR FIXED k.)

IF $n \leq k-1$ .. G HAS AT MOST $\binom{n}{2}$ EDGES.

FOR $n \geq k$ ASSUME THAT G IS A MAXIMAL GRAPH WITH NO K_k, I.E. ADDING ANY EDGE TO G CREATES A COPY OF K_k.

IN PARTICULAR G HAS A CLIQUE OF ORDER $k-1$.



$G' := G \setminus \{v_1, \dots, v_{k-1}\}$

CLIQUE

BY INDUCTION THE NUMBER OF EDGES OF G' IS AT MOST $\|T(n-k+1, k)\|$.

$\forall\, v \in V_{G'}$ HAS AT MOST $k-2$ NEIGHBOURS AMONG $v_1, \dots, v_{k-1}$ IN G. (OTHERWISE WE GET K_k)

$\Rightarrow$ THE NUMBER OF EDGES OF G IS AT MOST

$$\|T(n-k+1, k)\| + (n-k+1)\cdot(k-2) + \binom{k-1}{2} =$$
$$= \|T(n,k)\|. \blacksquare$$

06.11.2007 Tu
PETER ČERNO

EX: DM1012 KOMB. A GRAFY II

PROBLEM 1: FIND A FAMILY $\mathcal{F}$ OF s-SETS WITHOUT k-SUNFLOWER AND $|\mathcal{F}| = \binom{k+s-2}{s}$

PROBLEM 2: ... $|\mathcal{F}| = (k-1)^s$.

PROBLEM 3: $[n] = \{1, \dots, n\}$, $\mathcal{F} \subseteq 2^{[n]}$
PAIRWISE INTERSECTING $\Rightarrow$ $|\mathcal{F}| \leq 2^{n-1}$ BEST POSSIBLE.

1) $\binom{k+(s-1)-1}{(s-1)+1}$?

FOR $s=1$ $|\mathcal{F}| = k-1$ ✓

FOR $s=2$ $|\mathcal{F}| = \binom{k}{2}$ ✓

FOR $s=3$ $|\mathcal{F}| = \binom{k+1}{3}$

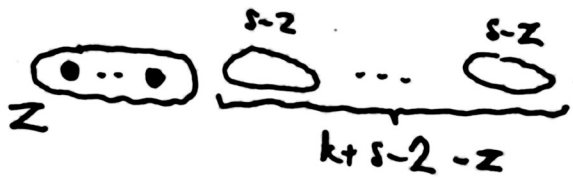
...

FOR s $|\mathcal{F}| = \binom{k+s-2}{s}$ $\mathcal{F} = \binom{[k+s-2]}{s}$

IF WE HAVE $A_1, \dots, A_m \in \binom{[k+s-2]}{s}$ SUNFLOWER

LET $Z = \bigcap A_i$.

$|Z| = z$

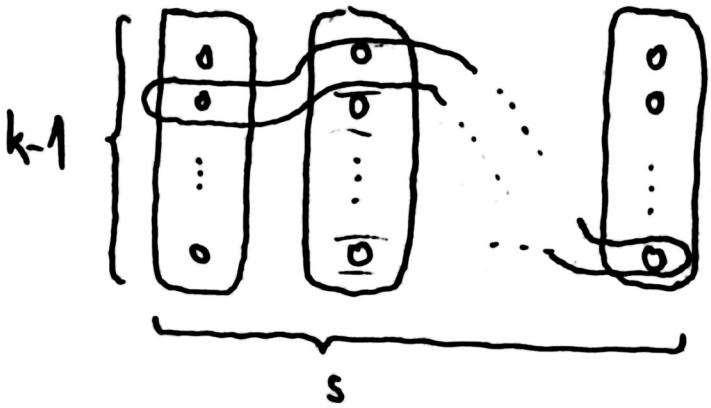


$$m \leq \frac{k+s-2-z}{s-z} = 1 + \frac{k-2}{s-z} \leq 1 + k-2 = \underline{\underline{k-1}}$$

$0 \leq z \leq s-1$

2) $|\mathcal{F}| = (k-1)^s$

$s = 1 \Rightarrow |\mathcal{F}| = k-1$ ✓

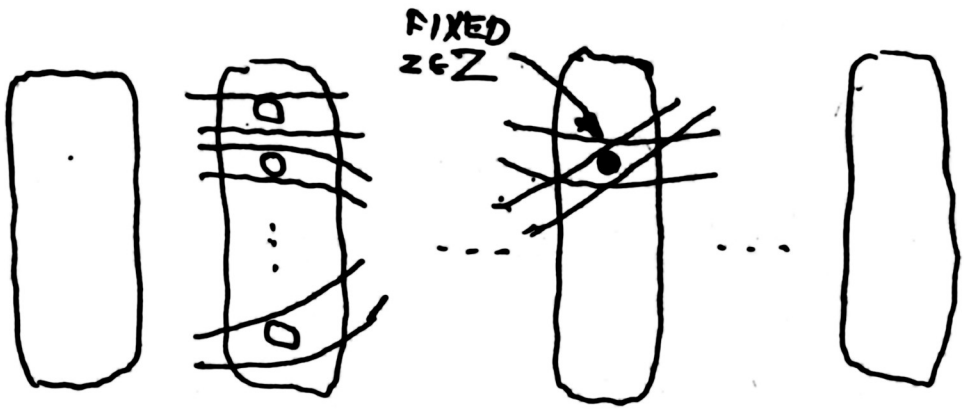


$|\mathcal{F}| = (k-1)^s$

$(k-1)^s$ POSSIBLE PATHS

$$\begin{matrix} a_{11} & a_{12} & \dots & a_{1s} \\ a_{21} & a_{22} & \dots & a_{2s} \\ \dots & & & \\ a_{k-1,1} & a_{k-1,2}, & \dots & a_{k-1,s} \end{matrix}$$

IF WE FIX SOME ELEMENTS TO BE IN THE CORE Z THEN IN EACH COLUMN CAN BE AT MOST ONE SUCH ELEMENT.



EVERY PATH CROSSES THIS ELEMENT z AND FOR EVERY $v \notin Z$ THIS VERTEX IS CROSSED BY AT MOST ONE PATH.

$\Rightarrow$ AT MOST $k-1$ DIFFERENT PATHS.

06.11.2007 Tu
PETER ČERNO

EX: DM1012 KOMB. A GRAFY II

3) $|\mathcal{F}| \leq 2^{n-1}$ IS BEST POSSIBLE

SIMPLE $\boxed{1}, 2, \ldots, n$

WE FIX THE NUMBER 1, SO EVERY $X \in \mathcal{F}$ CONTAINS 1. FOR THE OTHER NUMBERS $2, \ldots, n$ WE HAVE 2^{n-1} POSSIBILITIES.

SO $\mathcal{F} = \{X \in 2^{[n]} \mid 1 \in X\}$. $|\mathcal{F}| = 2^{n-1}$.

LET US ASSUME THAT $|\mathcal{F}| > 2^{n-1}$.
WE HAVE TO PROVE THAT THERE ARE TWO DISJOINT SETS.

LET US MAKE A TABLE:

$X \in 2^{[n-1]}$	$[n] \setminus X$
$\emptyset$	$\{1, 2, \ldots, n\}$
$\{1\}$	$\{2, \ldots, n\}$
$\vdots$	$\vdots$
$\{1, 2, \ldots, n-1\}$	$\{n\}$

IT HAS EXACTLY 2^{n-1} ROWS AND EVERY SUBSET $X \subseteq [n]$ IS IN THIS TABLE EXACTLY ONCE.

IF $n \notin X$ THEN X IS IN THE FIRST COLUMN
IF $n \in X$ THEN X IS IN THE SECOND COLUMN

WE HAVE $> 2^{n-1}$ SETS IN $\mathcal{F}$

$\Rightarrow$ SO AT LEAST TWO SETS FALL INTO ONE ROW AND WE HAVE TWO DISJOINT SETS. $\square$

PROBLEM 4: $G = (V, E)$, $|E| \geq (k-1)|V|$, $k \geq 2$

G CONTAINS ALL TREES WITH k EDGES AS SUBGRAPHS.

PROOF. $\sum \deg v = 2|E| \geq 2(k-1) \cdot |V|$

$\Rightarrow$ THE AVERAGE DEGREE OF G IS $2(k-1)$

WE CAN FIND A SUBGRAPH $G' \subseteq G$ SUCH THAT G' HAS MINIMUM DEGREE $2(k-1)$ (AND ALSO $2(k-1)+1$ VERTICES)

LET US PROVE THAT IF $\delta(G) \geq 2(k-1)$, $|G| \geq 2(k-1)+1$ ~~THERE IS~~ T IS A k-VERTEX TREE, THEN $T \subseteq G$.

PROOF INDUCTION BY k.

FOR $k=1$ .. ✓

LET $k > 1$. LET T BE A k-VERTEX TREE.

THIS TREE T HAS A LEAF w.

~~THE GRAPH G CONTAINS A VERTEX v WITH MINIMUM DEGREE $\geq 2(k-1)$~~

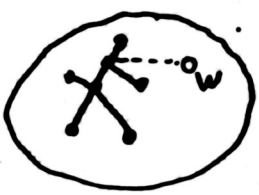
~~APPARENTLY $\delta(G \setminus v) \geq 2(k-1)-1 \geq 2(k-2)$~~

~~SO IT CONTAINS A TREE $T \setminus w$. (BY INDUCTION)~~

$T \setminus w$ IS ~~HAS~~ ~~$k-1$~~ $(k-1)$-VERTEX TREE.

BECAUSE $\delta(G) \geq 2(k-1) \geq 2(k-2)$

$T \setminus w$ IS IN THE GRAPH G



13.11.2007 Tu
PETER ČERNO

DM1012 KOMB. A GRAFY II

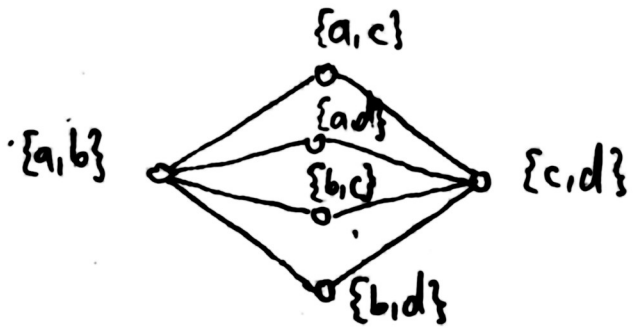
CHOOSEABILITY

DEF. (LIST COLORING): FOR A GRAPH G AND $L: V_G \to \mathcal{P}(C)$, COLOR G ($f: V_G \to C$) SO THAT
i) $\forall u \neq v$, $uv \in E_G$: $f(u) \neq f(v)$
ii) $\forall u \in V_G$: $f(u) \in L(u)$.

CHOOSEABILITY OF G $ch(G) = \chi_\ell(G) =$
$= \min\{k \in \mathbb{N} \mid \forall L : |L(u)| \geq k \ \exists L\text{-COLORING OF } G\}$

APPARENTLY $\chi(K_{2,4}) = 2$, BUT $ch(K_{2,4}) > 2$

$\boxed{\text{APPARENTLY } ch(G) \leq |V_G|}$



EXERCISE: SHOW THAT $ch(K_{2,4}) = 3$
EXERCISE: $\forall k \in \mathbb{N}$ $\exists$ BIPARTITE G : $\chi(G) = 2$ & $ch(G) \geq k$.

$\chi(G) \leq ch(G)$

PROOF. SUPPOSE $ch(G) = k$ $\Rightarrow$ CHOOSE $L(u) := \{1, \dots, k\}$ $\forall u \in V_G$.
$\Rightarrow$ WE GET A k-COLORING OF G $\Rightarrow$ $\chi(G) \leq k$. ∎

THEOREM: $ch(G) \leq \Delta(G) + 1$.

PROOF (BY INDUCTION ON $|V_G|$)

DEFINITION: A GRAPH IS CALLED k-DEGENERATE IF $\forall H \subseteq G \ \exists u \in V_H : \deg_H u \leq k$.

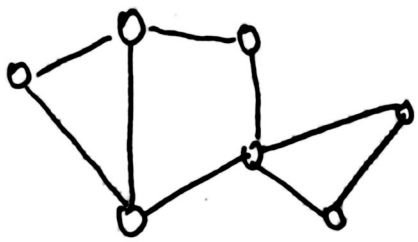
$\Leftrightarrow$ $\exists$ ORDERING $v_1, \dots, v_n$ SUCH THAT $\forall i \in [n] \ |N_G(v_i) \cap \{v_1, \dots, v_{i-1}\}| \leq k$.

EXAMPLE: TREES ARE 1-DEGENERATE

PLANAR GRAPHS ARE 5-DEGENERATE.

Δ-FREE PLANAR GRAPHS ARE 3-DEGENERATE

OUTERPLANAR GRAPHS ARE 2-DEGENERATE (EVERY VERTEX CAN BE SEEN FROM OUTSIDE OF THE GRAPH)



OBS. IF G IS k-DEGENERATE, THEN $ch(G) \leq k+1$.

PROOF: WE CHOOSE ORDERING $v_1, \dots, v_n$ AND COLOR FROM LEFT TO RIGHT.

13.11.2007 Ty
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DMI 012 KOMB. A GRAFY II

DEF. : LET $\mathcal{A}$ BE A CLASS OF GRAPHS

$$ch(\mathcal{A}) = \max_{G \in \mathcal{A}} ch(G)$$

1979 ERDŐS, RUBIN, TAYLOR
1976 VIZING

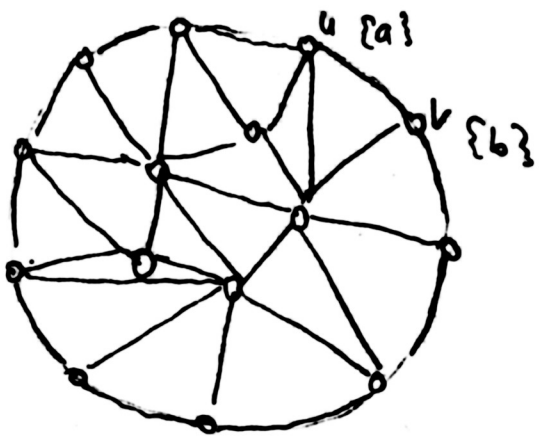
$4 \leq ch(\text{planar}) \leq 6$ (BECAUSE PLANAR GRAPHS ARE 5-DEGENERATE)

$$ch(K_4) \geq \chi(K_4) = 4$$

THOMASSEN : $ch(\text{planar}) \leq 5$
Voigt $ch(\text{planar}) \geq 5$

THEOREM : EVERY PLANAR GRAPH IS 5-CHOOSABLE.

THEOREM : IF G IS AN OUTER-TRIANGULATION



$a \neq b$

L IS AN ASSIGNMENT SUCH THAT

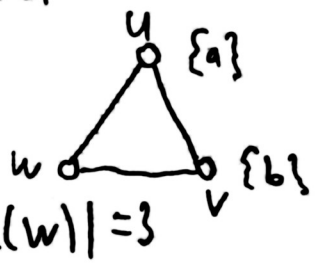
1) $\exists$ u,v NEIGHBOURS ON THE OUTERFACE $|L(u)| = |L(v)| = 1$, $L(u) \neq L(v)$

2) $\forall w \neq u, v$ ON OUTERFACE $|L(w)| = 3$

3) $\forall w$ INSIDE $|L(w)| = 5$

PROOF: (BY INDUCTION)

1)



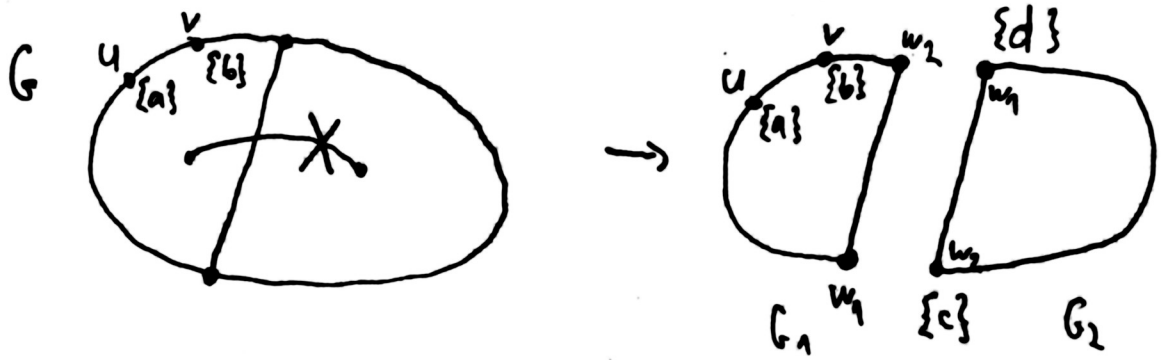
IS OK.

2) INDUCTION STEP FOR $n > 3$. $(|V_G| = n)$

LET G BE A GRAPH THAT IS OUTERTRIANGULATION

L SATISFIES ALL THE CONDITIONS.

a) THERE EXISTS A DIAGONAL IN G



WE CAN COLOUR G_1 BY INDUCTION.

IF $f(w_1) = c$, $f(w_2) = d$

$\Rightarrow$ WE CAN COLOUR G_2

$\Rightarrow$ WE GET COLOURING OF G.

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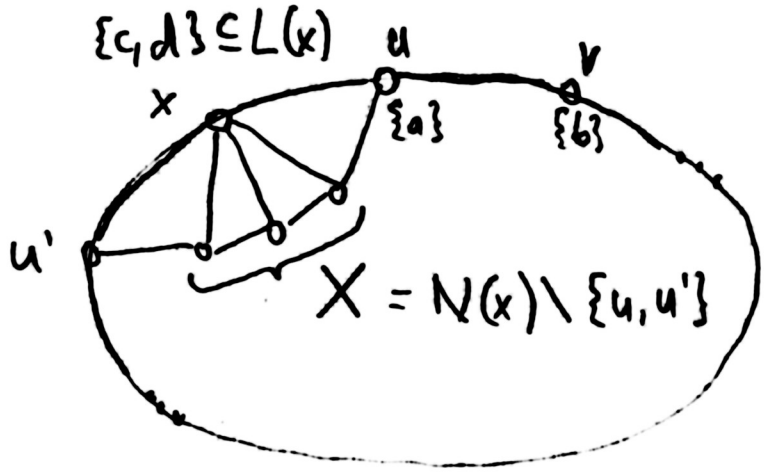
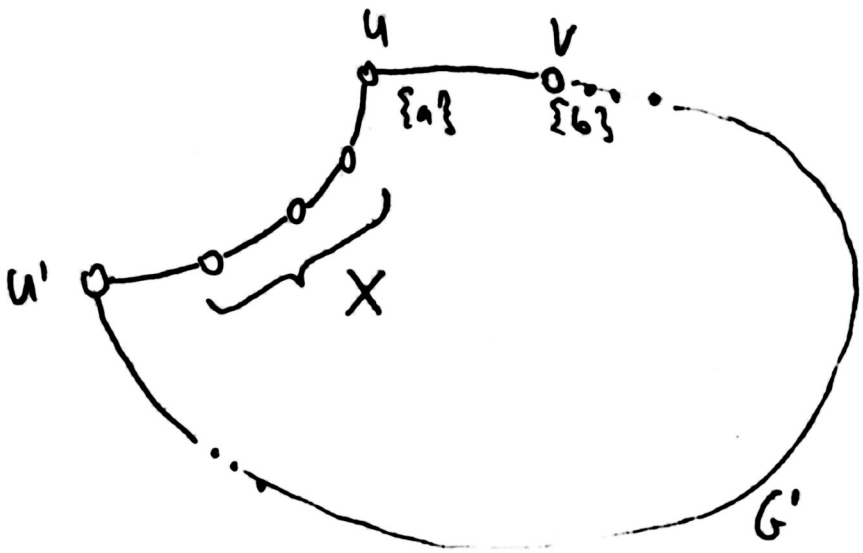
b) THERE IS NO DIAGONAL



IF WE DELETE x



WE DEFINE

$$L'(w) = \begin{cases} L(w) & w \notin X \\ L(w) \setminus \{c,d\} & w \in X \end{cases}$$

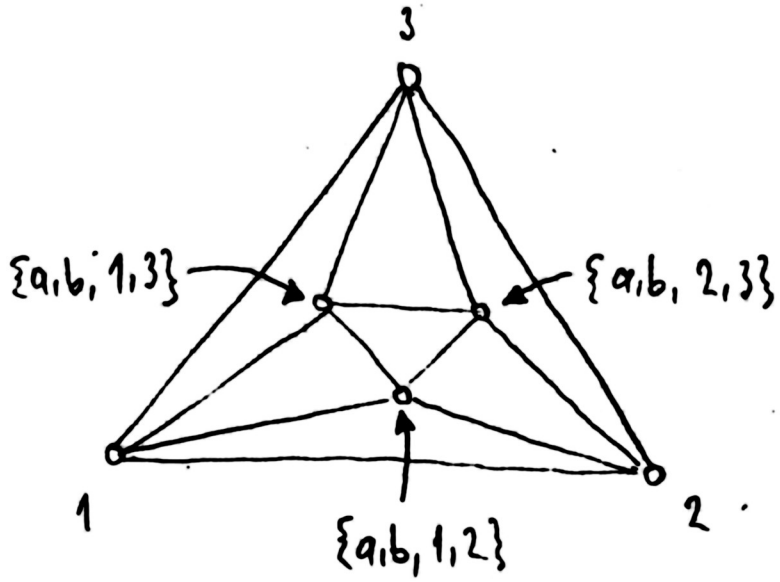
BY INDUCTION THERE EXISTS AN L'-COLORING f' OF G'.

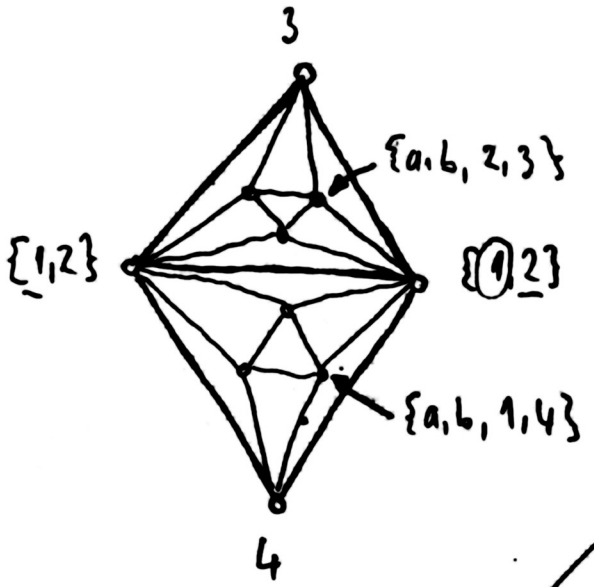
WE NEED TO DEFINE COLORING f OF G

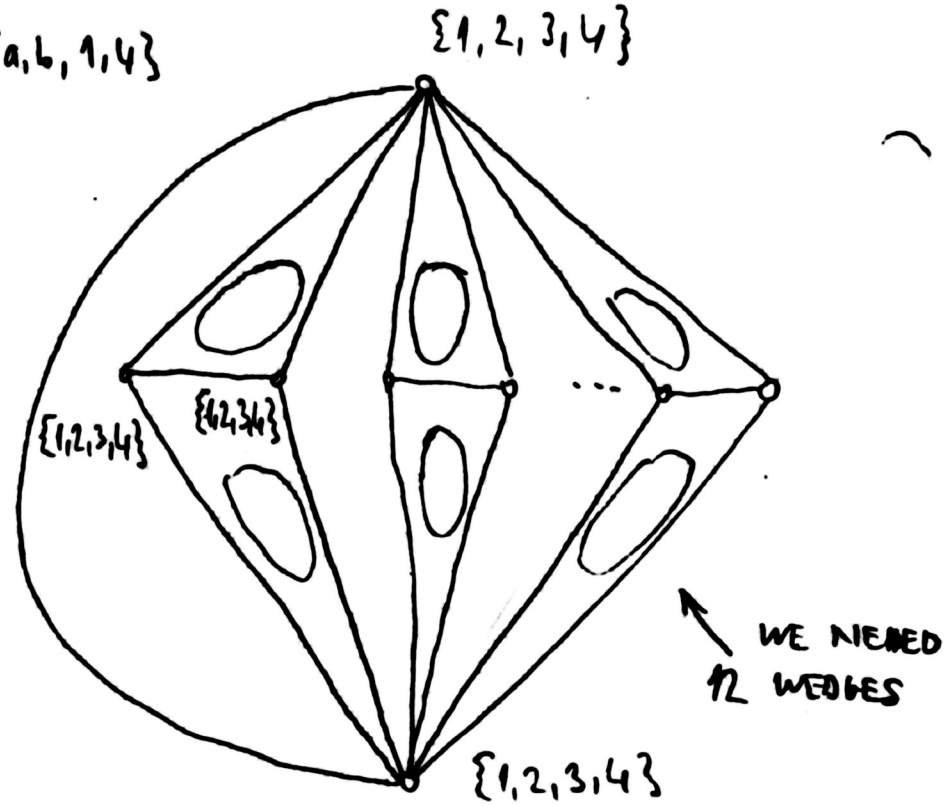
$$f(x) := \begin{cases} c & \text{IF } f(u') = d \\ d & \text{IF } f(u') = c \\ c \text{ OR } d & \text{IF } f(u') \notin \{c,d\} \end{cases}$$

IN X c AND d ARE FORBIDDEN COLORS

THEOREM : Ch(PLANAR GRAPHS) ≥ 5.







TOGETHER $12 \cdot 8 + 2 = 98$ VERTICES

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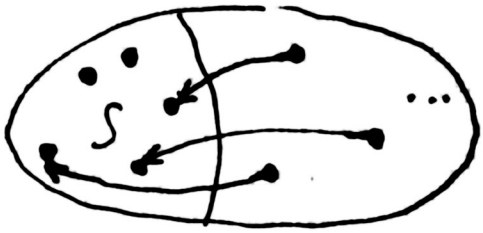
THEOREM: $3 \leq Ch(\text{planar bipartite}) \leq 4$

EXERCISE: SHOW THAT EVERY PLANAR BIPARTITE GRAPH IS 3-CHOOSABLE.

DEFINITION: A KERNEL IN A DIGRAPH D IS AN INDEPENDENT SET $S \subseteq V_D$ S.T. $\forall u \notin S \; \exists x \in S : \overrightarrow{ux} \in E_D$.



EXERCISE: SHOW A DIGRAPH WITH NO KERNEL.

EXERCISE: $\forall$ ORIENTATION OF A BIPARTITE GRAPH HAS A KERNEL.

THEOREM: IF G HAS AN ORIENTATION $\vec{G}$ S.T. 1) $\deg^+_{\vec{G}} u \leq k \quad \forall u$

2) EVERY $\vec{H} \subseteq \vec{G}$ HAS A KERNEL

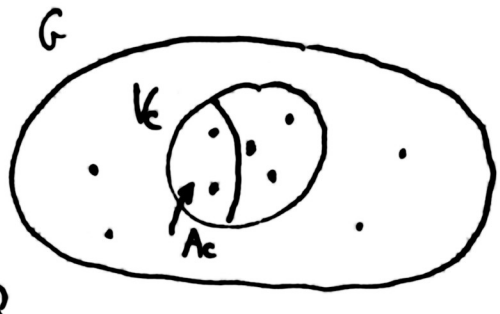
$\Rightarrow ch(G) \leq k+1$.

PROOF. WE ARE GIVEN G AND L, $|L(u)| \geq k+1$.

FOR $\forall$ COLOR $c \in \bigcup_{u \in V} L(u)$

$V_c \overset{\text{def}}{:=} \{ x \mid c \in L(x) \}$

$\vec{G}_c \overset{\text{def}}{:=} \vec{G}[V_c]$



LET A_c BE A KERNEL OF $\vec{G}_c$

FOR $x \in A_c$: $f(x) := c$ $L(x) := \{c\}$

FOR $y \in V_c \setminus A_c$: $L(y) := L(y) \setminus \{c\}$

CLAIM : THIS ALGORITHM PRODUCES AN L-COLORING OF G.

1) WHATEVER COLORING IT OUTPUTS, IT IS A CORRECT COLORING

2) WE HAVE TO PROVE THE LOOP INVARIANT OF THIS ALGORITHM : THAT NO LIST $L(u)$ BECOMES EMPTY.

IT IS CLEAR FROM $|L(u)| \geq k+1$

AND $\deg^+_{\vec{G}} u \leq k$

(WE DELETE AT MOST k-TIMES)

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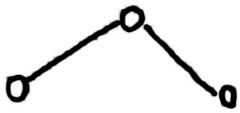
EX: DM1012 KOMB. A GRAFY II

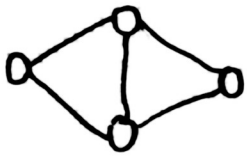
1) $ch(K_n - e)$

$ch(K_2 - e) = 1$

$ch(K_3 - e) = 2$ .....

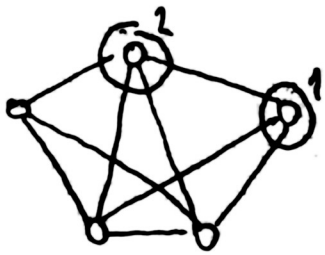


$3 \leq ch(K_4 - e) = 3$ .....



$K_4 - e$ IS 2-DEGENERABLE

$ch(K_5 - e) = 4$



$ch(K_n - e) = n-1$

PROOF: $K_n - e$ IS $n-2$-DEGENERABLE

LET $e = \{u, v\} \Rightarrow \deg_{K_n - e} u = n-2$

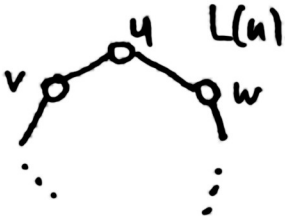
IF WE REMOVE THIS VERTEX u, WE GET K_{n-1} WHICH IS $n-2$-DEGENERABLE, BECAUSE $\Delta(K_{n-1}) = n-2$. ∎

2) $ch(C_n)$

IF $2 \mid n \Rightarrow \chi(C_n) = 2 \Rightarrow ch(C_n) \geq 2$.

$2 \nmid n \Rightarrow \chi(C_n) = 3 \Rightarrow ch(C_n) \geq 3$.

FOR $2 \nmid n$ WE HAVE $ch(C_n) = 3$ WHY?



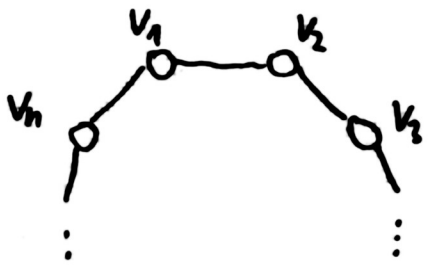
$L(u) = \{a, b, c\}$

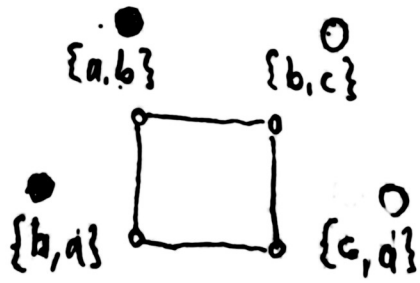
a) IF WE REMOVE u, WE GET 1-DEGENERABLE GRAPH

$\Rightarrow \exists$ L-COLORING f

b) $|L(u) \setminus \{f(v), f(w)\}| \geq 1$.

IF $2|n$



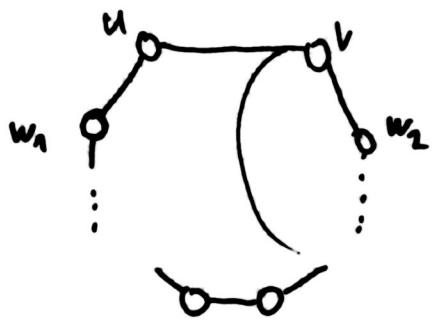


$n=2k$, INDUCTION BY k

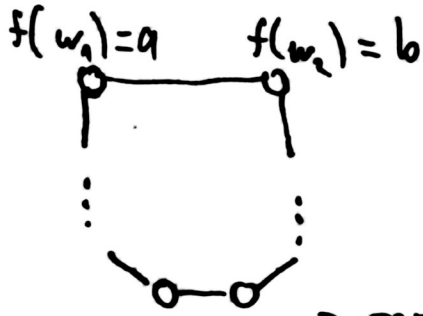
$k=1$

✓

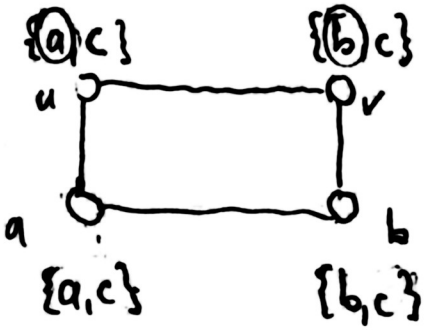
$k>1$



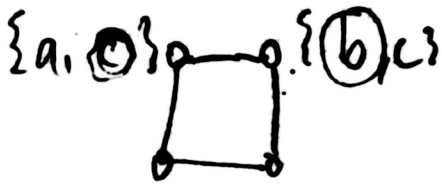
$\Rightarrow$



BY INDUCTION THERE EXISTS f



OOPS!



DOES NOT WORK

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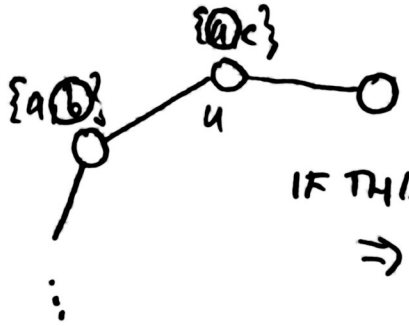
EX: DÚ 1012 KON. A GRAFY II

SUPPOSE WE HAVE 2 ADJACENT VERTICES WITH ATMOST 1 COMMON COLOR

WE PROCEED ALGORITHMICALLY:

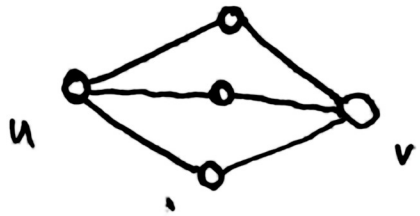
(OTHERWISE WE HAVE 2-COLORING)



IF THIS COLOR IS a
$\Rightarrow$ CHANGE COLOR OF u TO c. ■

WE PROCEED

3) $ch(K_{2,3}) = 2$

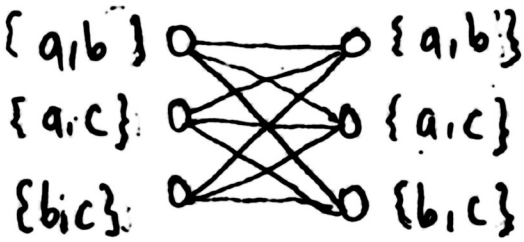


a) $L(u) \cap L(v) \neq \emptyset$
$\Rightarrow f(u) = f(v) \in L(u) \cap L(v)$
AND THEN IT IS SIMPLE

b) $L(u) \cap L(v) = \emptyset$
$L(u) = \{a,b\}$, $L(v) = \{c,d\}$
WE HAVE 4 POSSIBILITIES FOR $(f(u), f(v))$. □

4) $ch(K_{2,33}) = 3$ IT CONTAINS $K_{2,4}$ AND IS 2-DEGENERABLE

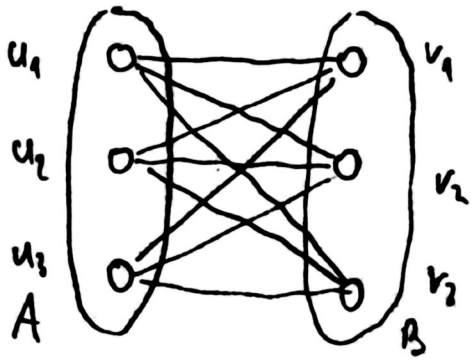
5) $2 \leq ch(K_{3,3}) \leq 4$ $\chi(K_{3,3}) = 2$, $K_{3,3}$ IS 3-DEGENERABLE.



NA ĽAVO 2 FARBY z $\{a,b,c\}$
NA PRAVO TIEŽ
$\Rightarrow$ PRIENIK OBSAHUJE NUTNE ASPOŇ 1 FARBU

$\Rightarrow 3 \leq ch(K_{3,3})$

LET US PROVE $ch(K_{3,3}) = 3$



a) IF A HAS 1 OR 2 COMMON COLORS
$\Rightarrow$ IT IS EASY

b) IF A AND B HAVE NO COMMON COLOR THE WE CHOOSE IN A SUCH COMBINATION $(f(u_1), f(u_2), f(u_3))$ THAT IS DIFFERENT FROM ALL $L(v_1), L(v_2), L(v_3)$

6) $3 \leq ch(K_{3,3333}) \leq 4$ IT IS 3-DEGENERABLE + IT CONTAINS $K_{3,3}$

LET US PROVE $ch(K_{3,3333}) = 4$

$\{a,b,c\}$ O
$\{d,e,f\}$ O
$\{g,h,i\}$ O

O $\{a,d,g\}$
O $\{a,d,h\}$
⋮
O $\{c,f,i\}$
⋮
O

TOGETHER $3^3 = 27$

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EX: DM1012 KOMB. A GRAFY II

7) $G = (V, E)$, $\bar{G} = (V, \binom{V}{2} \setminus E)$

PROVE THAT $ch(G) + ch(\bar{G}) \leq n+1$ $\quad n = |V_G|$

(ALSO PROVE $\chi(G) + \chi(\bar{G}) \leq n+1$)

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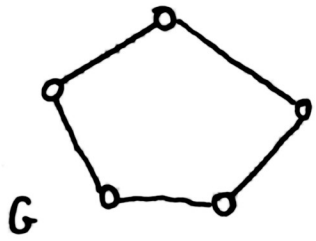
DM1012 KOMB. A GRAFY II

EDGE-COLORING OF G IS A COLORING OF EDGES SUCH THAT NO TWO EDGES SHARING A VERTEX GET THE SAME COLOR. THE SMALLEST NUMBER OF COLORS NEEDED TO ACHIEVE THIS GOAL IS CALLED THE CHROMATIC INDEX OF G AND DENOTED BY $\chi'(G)$.

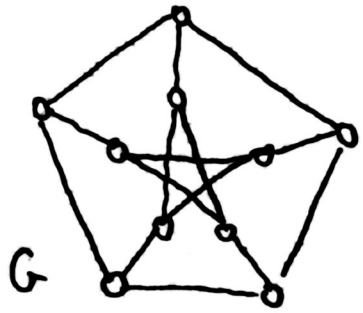
$\Delta(G) \leq \chi'(G)$.

THEOREM (VIZING): IF G IS SIMPLE, THEN $\chi'(G) \leq \Delta(G)+1$.



G

$\Delta(G) = 2$
$\chi'(G) = 3$

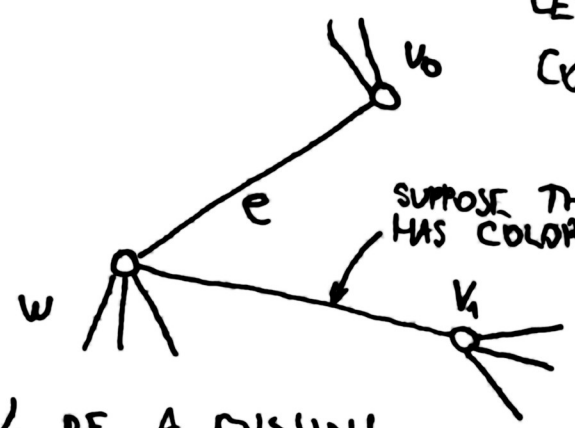


G

$\Delta(G) = 3$
$\chi'(G) = 4$

PROOF. LET G BE THE SMALLEST COUNTER-EXAMPLE. CHOOSE ARBITRARY EDGE $e \in G$.

$G \setminus e$ HAS AN EDGE-COLORING WITH AT MOST $\Delta(G \setminus e)+1$ COLORS.

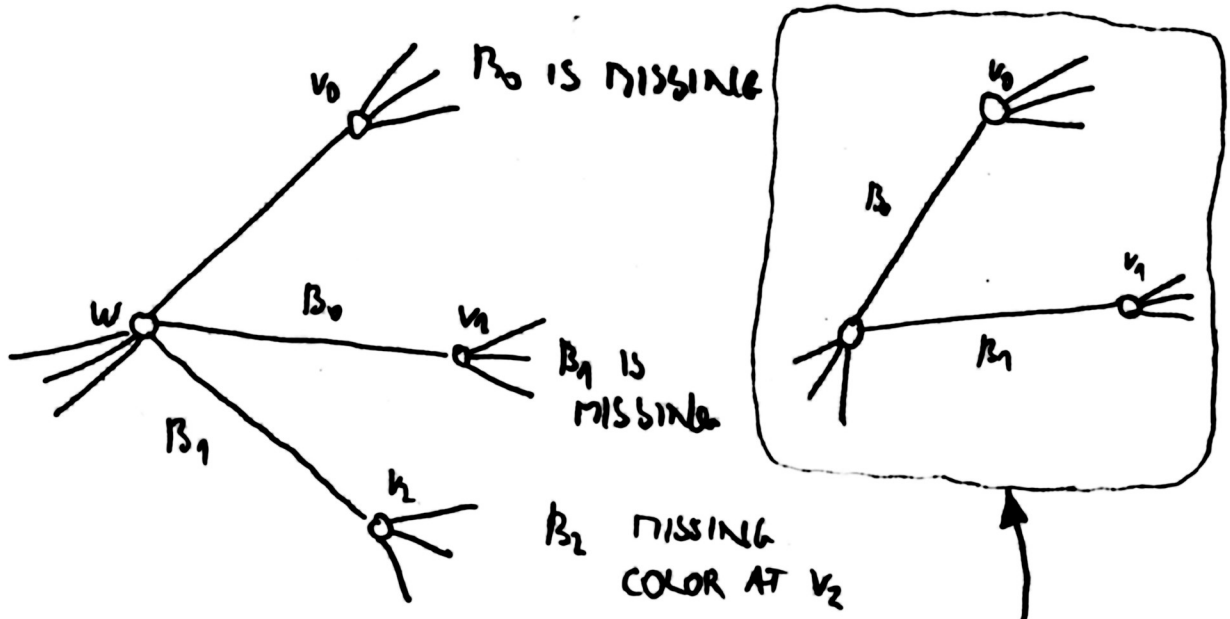


LET β_0 BE A MISSING COLOR AT VERTEX v_0

SUPPOSE THAT wv_1 HAS COLOR β_0

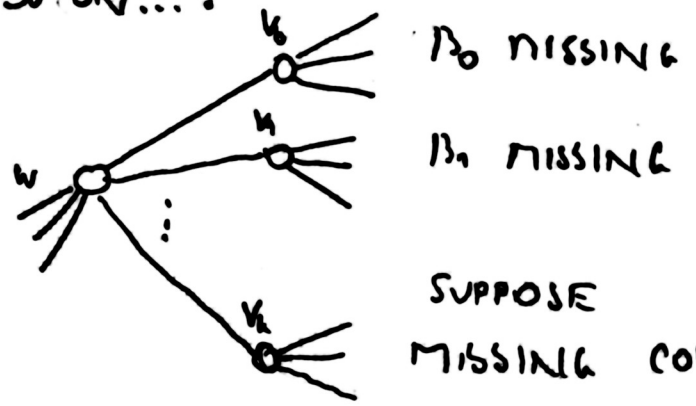
LET β_1 BE A MISSING COLOR AT v_1

LET α BE A MISSING COLOR AT VERTEX w

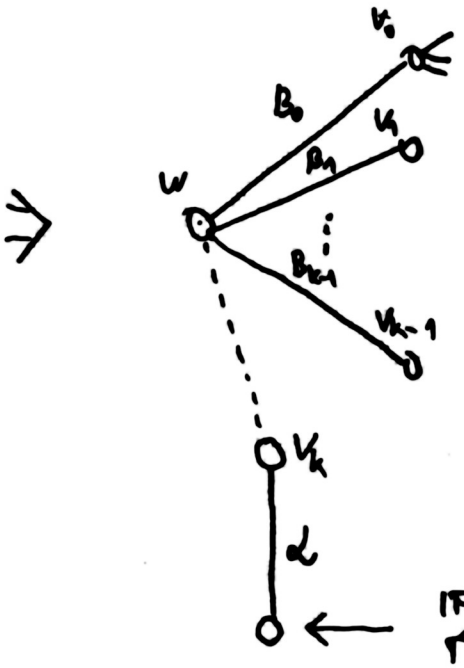


SUPPOSE β_1 EXIST AT w. OTHERWISE

AND SO ON... :



SUPPOSE
MISSING COLOR $= \beta_i$ $i < k$.



IF β_i IS MISSING THERE, WE CAN COLOR THIS EDGE WITH β_i AND THE PREVIOUS WITH α

SUPPOSE β_i IS NOT MISSING

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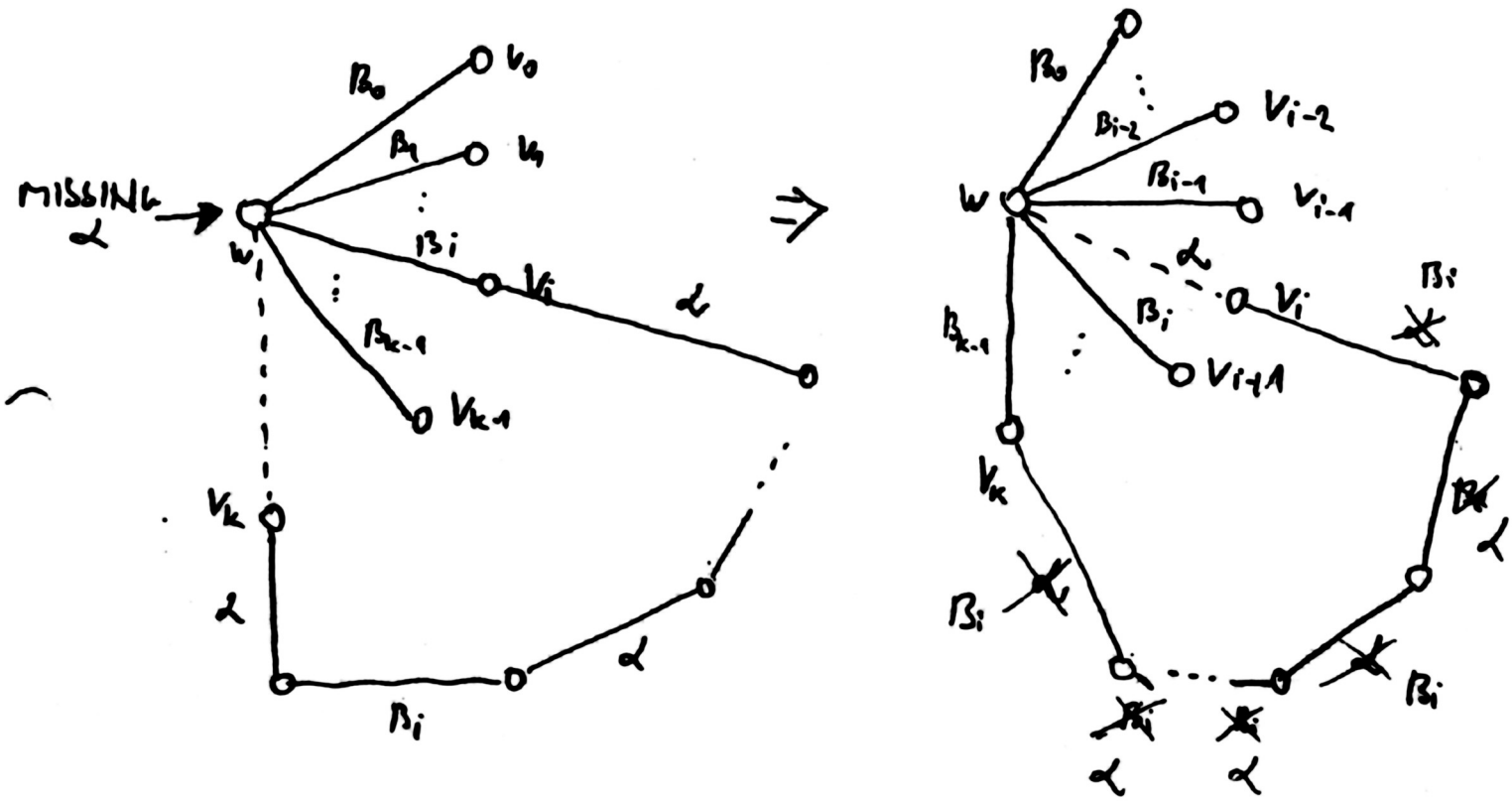
IF WE PROCEED WE GET A PATH:



LIST COLORING (CHOOSABILITY) - THE LIST CHROMATIC NUMBER (CHOOSABILITY) OF A GRAPH G IS THE SMALLEST k SUCH THAT IF EVERY VERTEX $v \in V_G$ HAS A LIST $L(v)$ OF k AVAILABLE COLORS, THEN THERE EXISTS A PROPER COLORING c SUCH THAT $c(v) \in L(v)$.

THIS LIST CHROMATIC INDEX IS DEFINED ANALOGOUSLY.

CONJECTURE: FOR EACH GRAPH (SIMPLE) G:
$\chi'(G) = \chi'_\ell(G)$.

THEOREM (GALVIN): THE LIST CHROMATIC INDEX OF A BIPARTITE GRAPH IS EQUAL TO $\Delta(G)$.

THEOREM: FOR EVERY BIPARTITE GRAPH G, IT HOLDS THAT $\chi'(G) = \Delta(G)$.

PROOF (EXERCISE)

IF G HAS AN ORIENTATION SUCH THAT EVERY INDUCED SUBGRAPH H HAS A KERNEL, THEN $\chi_\ell(G) \leq \Delta^+(G)+1$

THE MINIMUM OUT-DEGREE

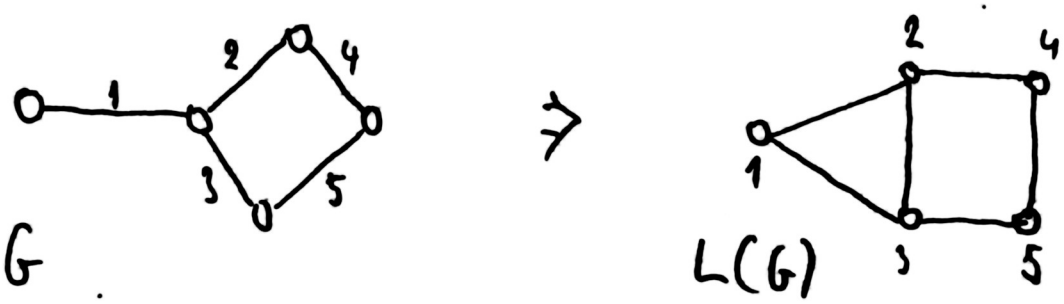
A KERNEL OF A DIRECTED GRAPH IS A SET $U \subseteq V_G$ SUCH THAT U IS INDEPENDENT AND THERE IS AN EDGE FROM EVERY VERTEX $v \notin U$ TO A VERTEX IN U.

PROOF OF GALVIN'S THEOREM:

LINE-GRAPH OF G IS THE GRAPH $L(G)$

VERTICES OF $L(G)$ ... EDGES OF G

EDGES $e_1e_2 \in E_{L(G)}$ ... $e_1 \cap e_2 \neq \emptyset$.



BIPARTITE GRAPH G HAS EDGE-COLORING WITH Δ COLORS. $(1, 2, 3, \ldots, \Delta(G))$

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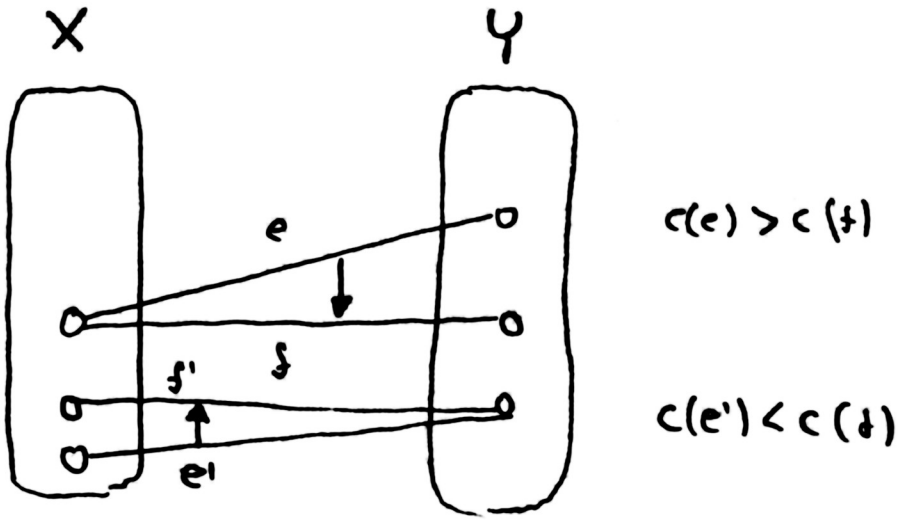


IF e AND f SHARE A VERTEX IN X
$\Rightarrow$ $e \longrightarrow f$ IF $c(e) > c(f)$ IN $L(G)$.

IF e' AND f' SHARE A VERTEX IN Y
$\Rightarrow$ $e' \longrightarrow f'$ IF $c(e') < c(f)$.

WE HAVE TO FIND OUT:
1) MAXIMUM OUT-DEGREE
2) EVERY INDUCED SUBGRAPH HAS A KERNEL

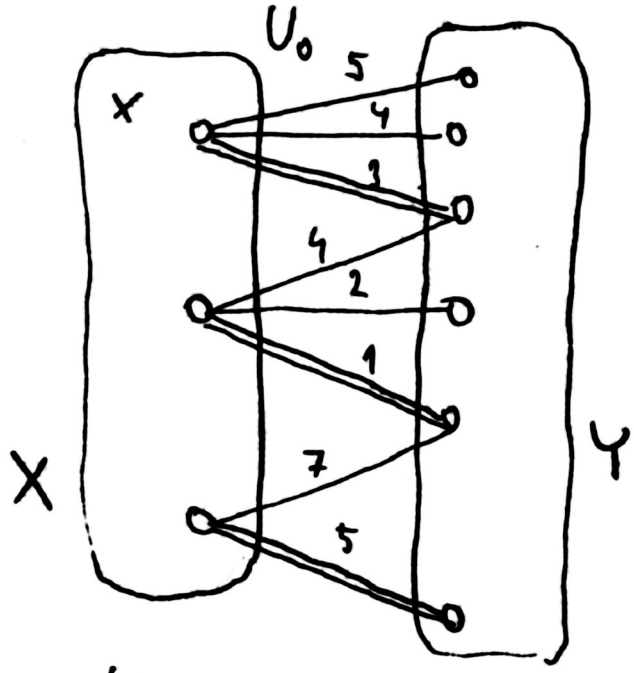
1) IF e HAS A COLOR K $\Rightarrow$ OUT-DEGREE $\leq$
$\leq K-1 + (\Delta - K) = \Delta - 1$.

2) INDUCTION BY # VERTICES OF A SUBGRAPH OF $L(G)$
(# CHOSEN EDGES OF G)

FIX A SET U_0 OF EDGES OF G.

1) IF $|U_0| = 1, 2 \Rightarrow$ ✓

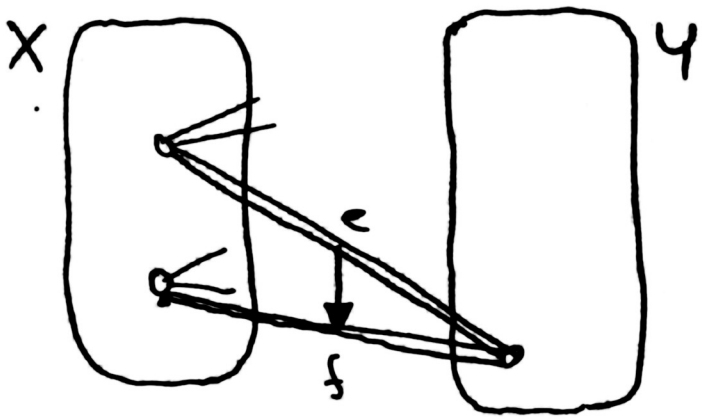
2) LET $U = \{e \in U_0$ AND e IS AN EDGE WITH SMALLEST COLOR AT A VERTEX $x \in X$ OUT OF ALL ~~COLORS~~ EDGES OF U_0 INCIDENT WITH x



IF U IS MATCHING $\Rightarrow$ WE ARE DONE

U IS INDEPENDENT AND

SUPPOSE U IS NOT A MATCHING



SUPPOSE $c(f) > c(e)$.

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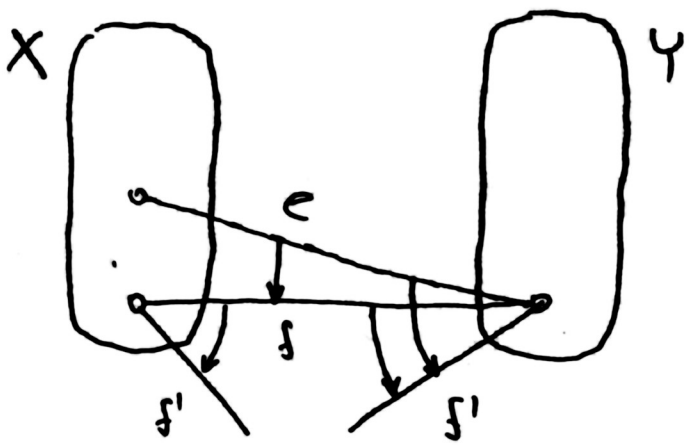
DMI 012 KOMB. A GRAFY II

BY INDUCTION $U_0 \setminus \{e\}$ HAS A KERNEL U'.

IF $f \in U'$ ... $\Rightarrow$ OK SINCE U' IS A KERNEL WITH RESPECT TO U_0.

IF $f \notin U'$ ... $\Rightarrow$ THERE EXISTS $f' \in U'$ AND $f \to f'$ IN $L(G)$.

f AND f' SHARE A VERTEX IN G $\in X$, $\in Y$



IF THIS VERTEX $\in Y \Rightarrow e \to f \wedge f \to f' \Rightarrow e \to f'$.

IF THIS VERTEX $\in X$, THEN $f \to f'$ WHICH IS NOT POSSIBLE, BECAUSE OF CHOICE OF U.

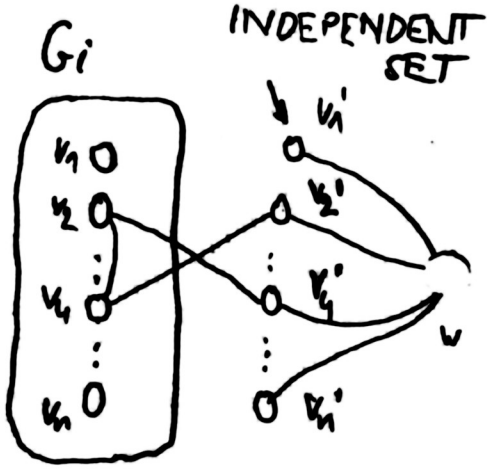
THEOREM : $\omega \leq \chi \leq \Delta + 1$

$\leq \Delta$ IF G IS CONNECTED AND $\neq C_{2\ell+1}$, $\neq K_n$

THEOREM : THERE EXISTS A GRAPH WITH $\omega = 2$ AND χ CAN BE LARGE.

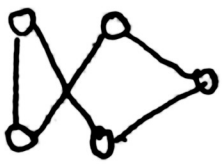
$G_2 =$

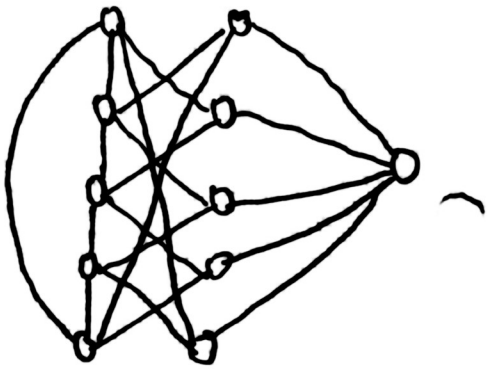
$G_i =$

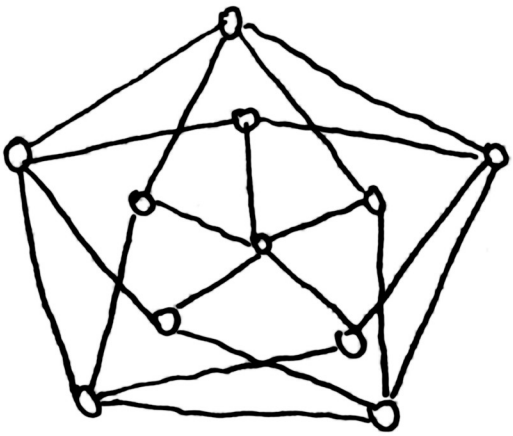
$G_{i+1} =$ 

$v_j' v_k \in E_{i+1} \Leftrightarrow v_j v_k \in E_i$

$G_2 =$

$G_3 =$ 

$G_4 =$ 

$=$ 

1) G_i HAS NO TRIANGLES

2) $\chi(G_i) = i$

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1) INDUCTION BY i

IF G_{i+1} HAS Δ ABC

a) $w \notin \{A, B, C\}$

b) $|\{A, B, C\} \cap \{v_1', \dots, v_n'\}| \leq 1$

c) $\{A, B, C\} \subseteq \{v_1, \dots, v_n\}$

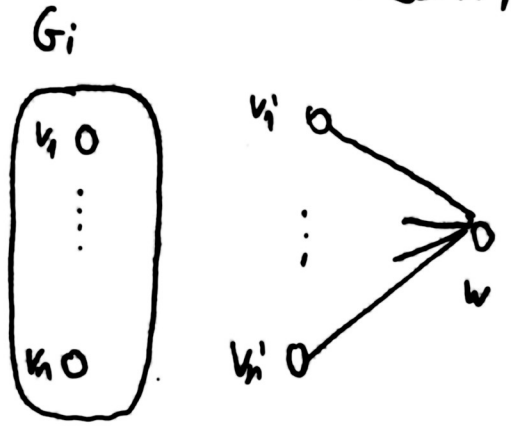
$\Rightarrow A = v_{j_1}', B = v_{j_2}, C = v_{j_3}$

$\Rightarrow v_{j_1} v_{j_2} v_{j_3}$ IS A Δ IN G_i. ↯

2) FIX COLORING G_i WITH i COLORS AND USE IT ON $v_1', \dots, v_n'$. ASSIGN w A NEW COLOR.

ASSUME THAT $\chi(G_{i+1}) \leq i$ (FOR THE SAKE OF CONTR.)

IE $\chi(G_{i+1}) = i$ FIX COLORING c OF G_{i+1}



$$c'(v_i) = \begin{cases} c(v_i') & \text{IF } c(v_i) = c(w) \\ c(v_i) & \text{OTHERWISE} \end{cases}$$

$\Rightarrow$ APPARENTLY c' IS COLORING OF G_i WITH $i-1$ COLORS.

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EX: DMI 012 KOMB. A GRAFY II

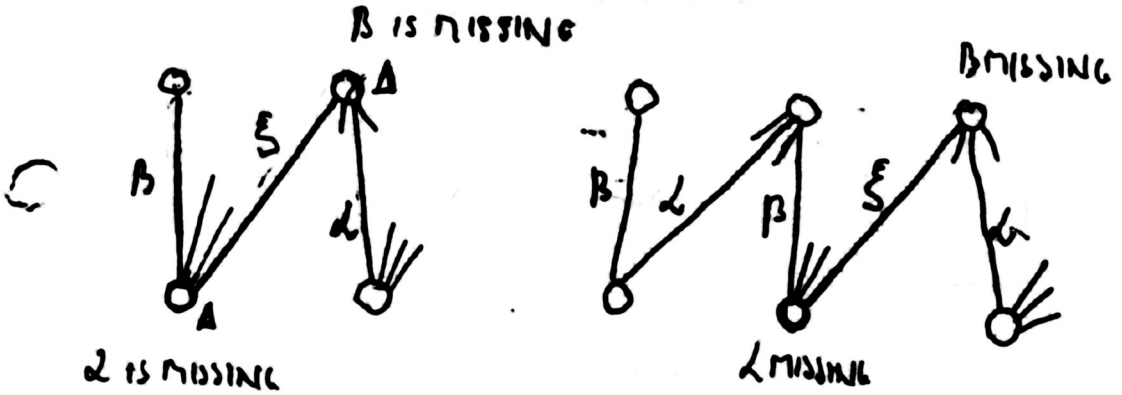
1) PROVE THAT EVERY BIPARTITE GRAPH WITH MAXIMUM DEGREE Δ IS Δ-EDGE-COLORABLE.

HINT: FIRST PROVE THIS FOR Δ-REGULAR BIPARTITE GRAPHS.

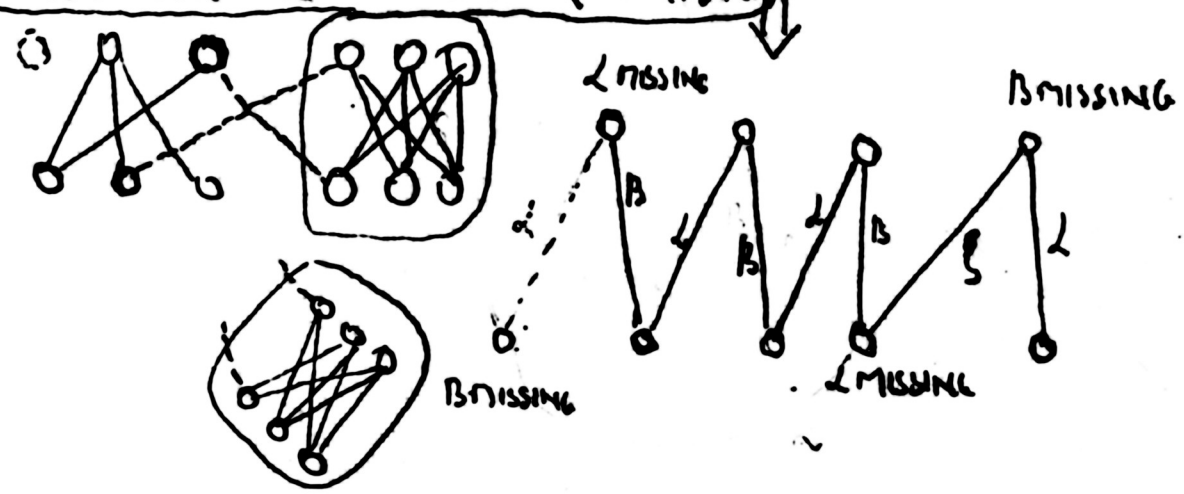
2) PROVE THAT THE FOUR COLOR THEOREM IS EQUIVALENT TO THE STATEMENT ASSERTING THAT EVERY CUBIC BRIDGELESS PLANAR GRAPH IS 3-EDGE-COLORABLE.

3) SHOW THAT EVERY TRIANGULATION OF THE PLANE WITH ALL VERTICES OF EVEN DEGREE IS 3-COLORABLE

HINT TO 2), 3): DUALIZE!

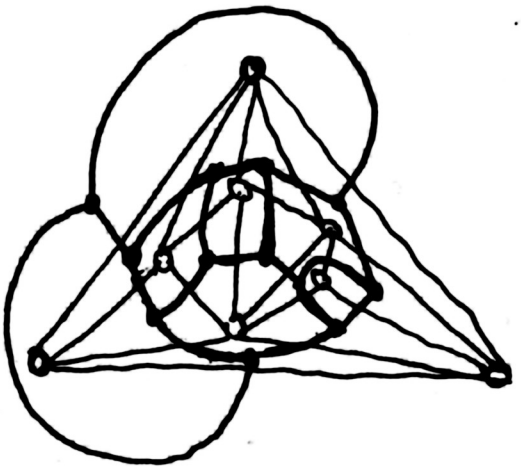


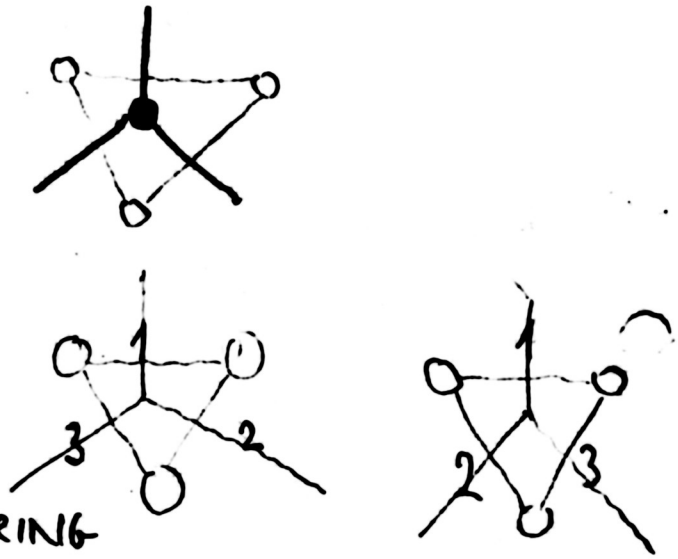
INDUCTION: Δ-REG. BIP. G. \ PERF. MATCH.



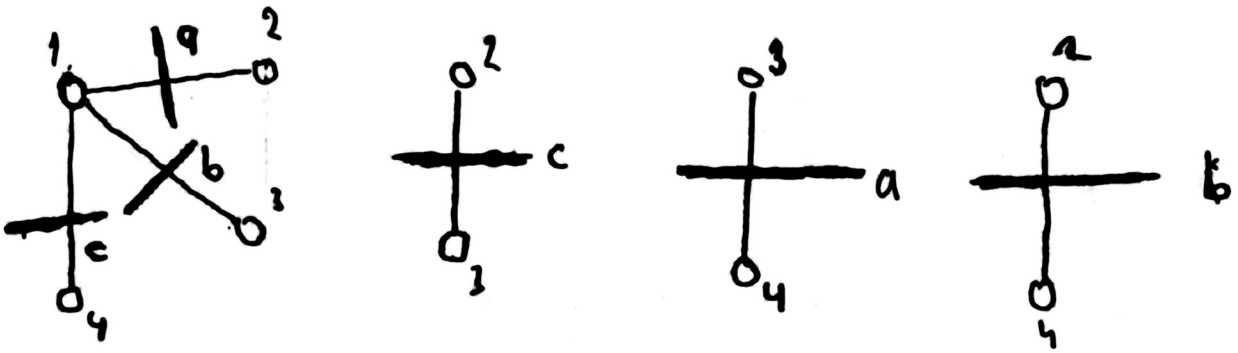
21

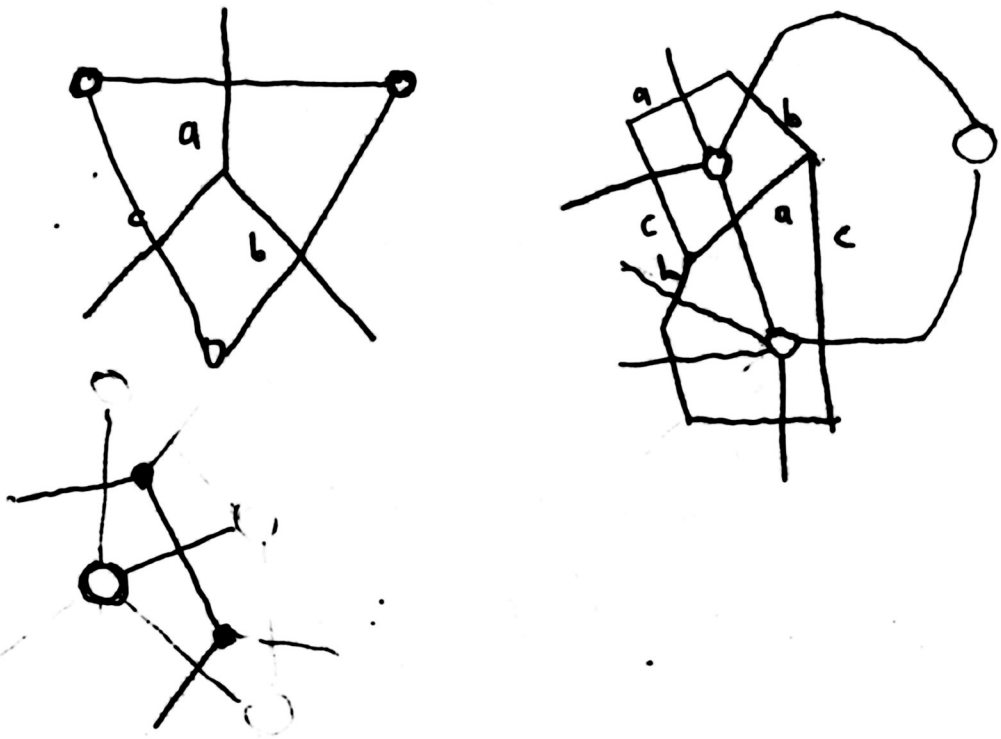
G IS PLANAR $\Rightarrow$ TRIANGULATION





EASY IMPLICATION :
4 COLOR TH. $\Rightarrow$ EDGE-COLORING





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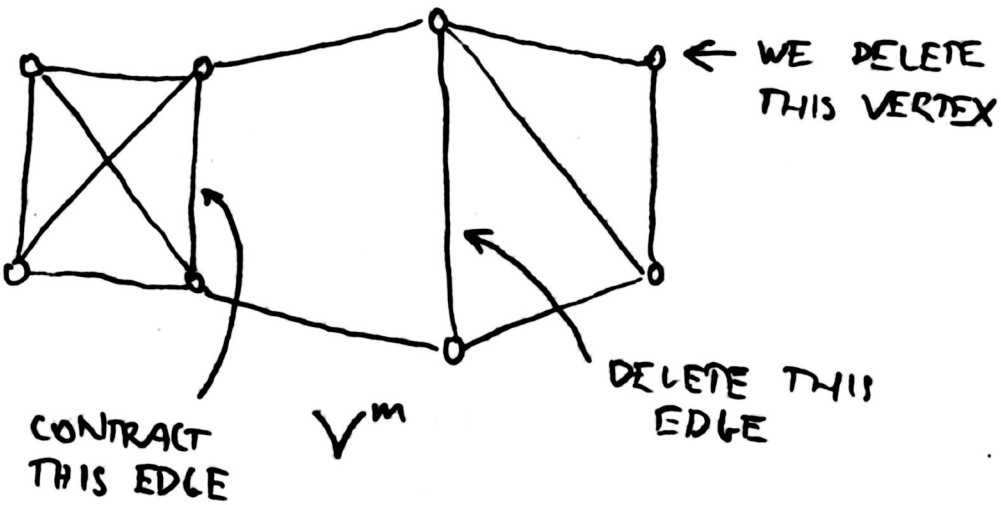
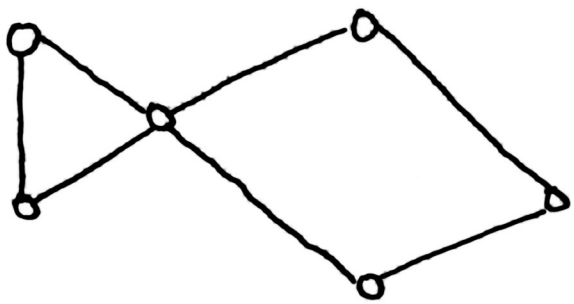
GRAPH MINORS AND (INTRODUCTION TO) TREE-WIDTH

DEF. $H <_m G$ IF A GRAPH ISOMORPHIC TO H CAN BE OBTAINED FROM G BY

a) DELETING VERTICES
b) DELETING EDGES
c) CONTRACTING EDGES



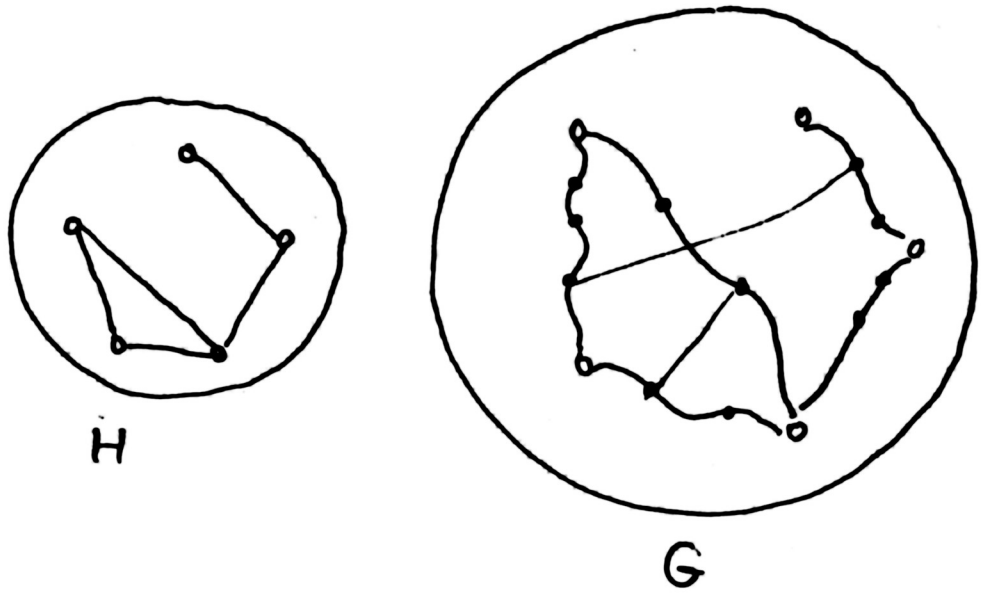
V^m



APPARENTLY $C_k <_m C_n$ IF $k < n$.

$G <_m K_n$, WHERE $n = |V_G|$.

LEMMA 1: IF G CONTAINS A SUBDIVISION OF H, THEN $H <_m G$.



FIRST WE DELETE ALL THE VERTICES THAT ARE NOT ON THE PATHS

SECOND WE DELETE EDGES

AND THEN CONTRACT EDGES ON PATHS

$\rightarrow$ WE GET H.

LEMMA 2: IF $\Delta(H) \leq 3$ THEN $H <_m G \Leftrightarrow$ G CONTAINS A SUBDIVISION OF H.

NOTE: WE CAN PRESCRIBE THE ORDER OF OPERATIONS: a), b), c)

LEMMA 3: $H <_m G \Leftrightarrow \exists V_u \subseteq V(G) \quad u \in V(H)$

SUCH THAT i) $\forall u \neq v \; : V_u \cap V_v = \emptyset$

ii) $\forall u \; : G[V_u]$ IS CONNECTED

iii) $\forall uv \in E(H) \; \exists x \in V_u, y \in V_v : xy \in E(G)$

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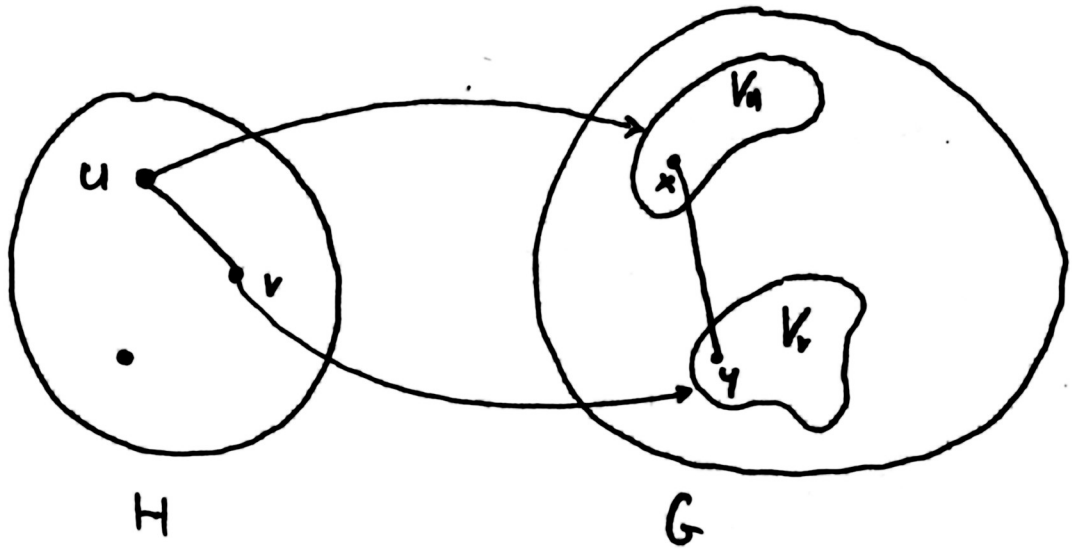


PROOF (OF LEMMA 3):

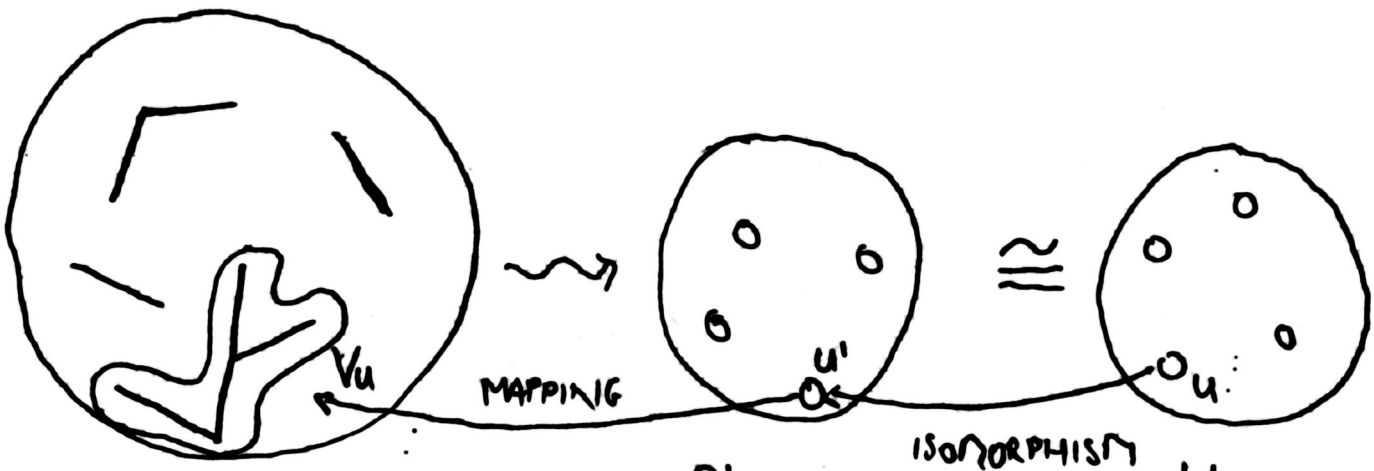
$\Leftarrow$) 1) DELETE ALL THE VERTICES THAT DO NOT BELONG TO $\bigcup V_u$

2) CONTRACT ALL THE SUBGRAPHS V_u TO ONE VERTEX

3) DELETE ALL THE EDGES THAT WE DO NOT WANT

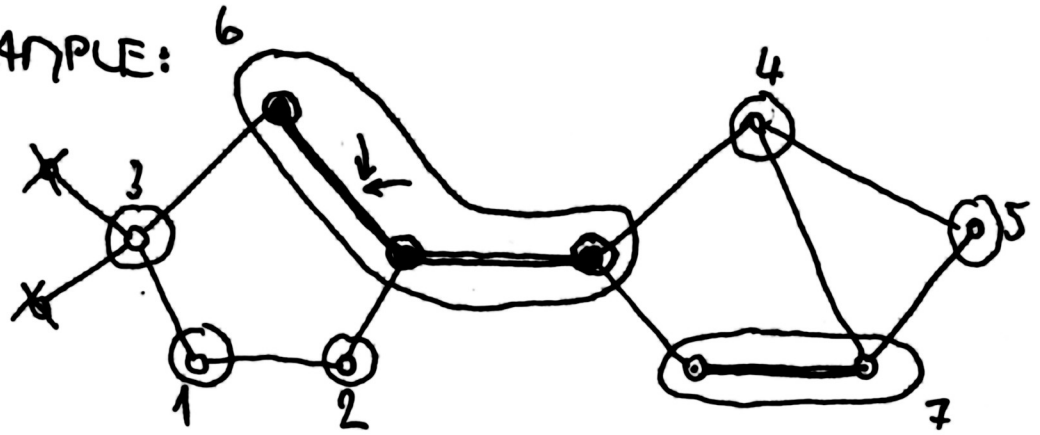
$\Rightarrow$) IF $H \prec_m G$

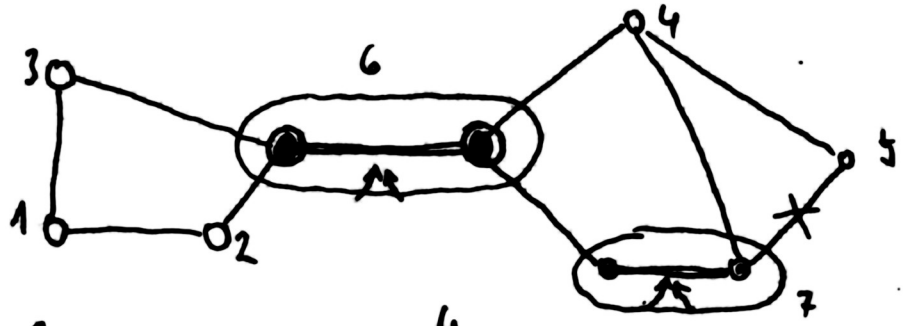
WE JUST KEEP THE TRACK OF <u>EDGE CONTRACTING</u>.

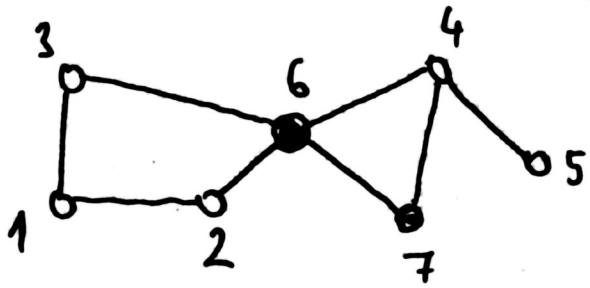


G AND CONTRACTED EDGES NOTED

EXAMPLE:

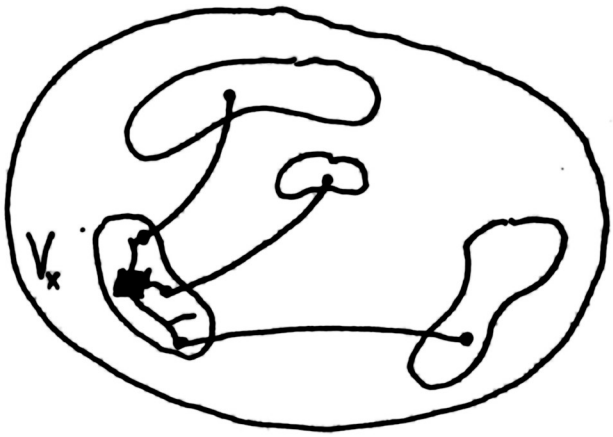




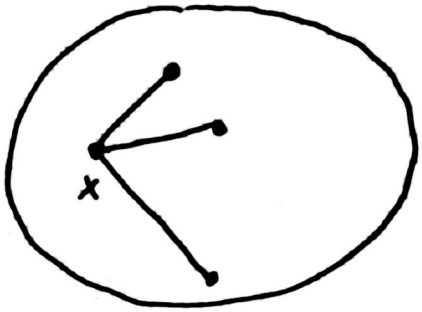


PROOF (OF LEMMA 2) $\Leftarrow$) WE KNOW

$\Rightarrow$) BY LEMMA 3:



G



H

IMPORTANT SQUARE VERTEX

$\Rightarrow$ WE CAN EASILY FIND A SUBDIVISION OF H.

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EXERCISE: SHOW THAT LEMMA 2 IS NOT TRUE IF $\Delta(H) \geq 4$.

EXERCISE: SHOW THE MINOR VERSION OF KURATOWSKI: G IS PLANAR $\Leftrightarrow$ $\neg K_5 <_m G \wedge \neg K_{3,3} <_m G$.

HADWIGER CONJECTURE:

$$\forall k : \chi(G) > k \Rightarrow K_{k+1} <_m G$$

FOR $k=1$: $\chi(G) > 1 \Rightarrow \underbrace{E(G) \neq \emptyset}_{K_2 <_m G}$

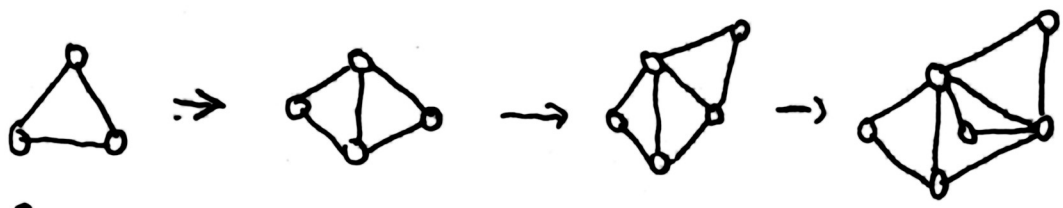
FOR $k=2$: $\chi(G) > 2 \Rightarrow C_{2k+1} \subseteq G \Rightarrow$
$C_3 <_m G \equiv K_3 <_m G$

FOR $k=3$: $\chi(G) > 3$ ... WE DO NOT KNOW WHAT THIS MEANS

WHAT DOES $K_4 <_m G$ MEAN?

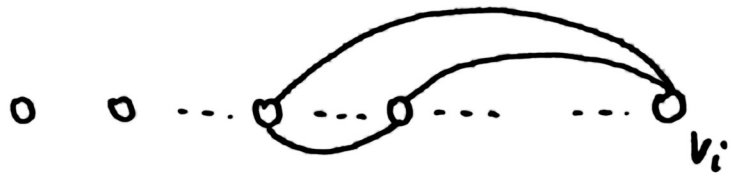
WE WANT TO PROVE $\neg K_4 <_m G \Rightarrow \neg \chi(G) > 3$.

DEFINITION: 2-TREE



G IS A 2-TREE IF $|V(G)| \geq 3$ AND $\exists$ ORDERING OF VERTICES: $v_1, v_2, v_3, \ldots, v_n$ SUCH THAT $v_1 v_2 \in E(G)$ $\forall i = 3, \ldots, n$.

IF WE LOOK AT THE NEIGHBOURHOOD OF v_i IN THE GRAPH $G[\{v_1, v_2, \dots, v_i\}]$, WE GET:



$$|N_{G[v_1,\dots,v_i]}(v_i)| = 2$$

$$G[N_{G[v_1,\dots,v_i]}(v_i)] \cong K_2$$

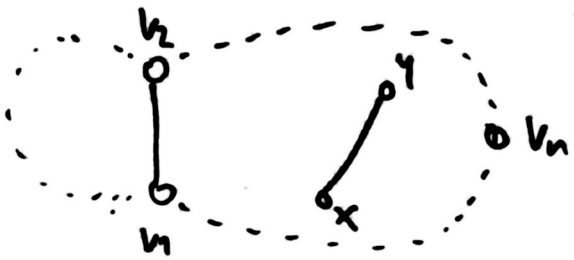
THE SEQUENCE $v_1, v_2, \dots, v_n$ IS CALLED GROWING OF G FROM $v_1 v_2$.

LEMMA 4: A 2-TREE CAN BE GROWN FROM ANY OF ITS EDGES.

PROOF: $v_1 v_2 \dots v_n$ (PROOF BY INDUCTION)

1) $|V(G)| = 3$. IS APPARENTLY TRUE

2) $|V(G)| > 3$.



a) $v_n \notin xy$. $v_1, \dots, v_{n-1}$ IS ALSO A 2-TREE G'

BY INDUCTION $\exists$ $xy \dots$ GROWING OF G'

AND APPARENTLY $xy \dots v_n$ IS GROWING OF G.

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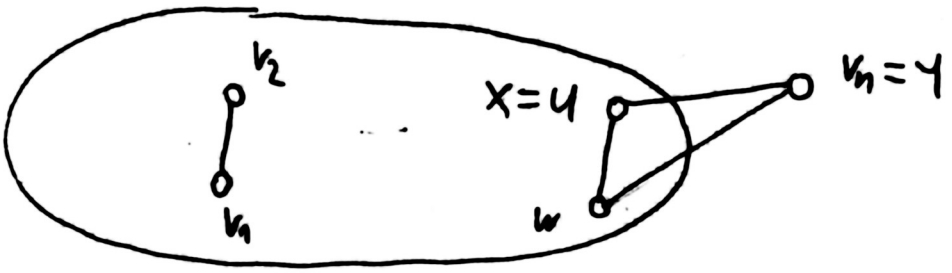
b) $v_n \in xy$



$G' = G \setminus v_n$ (BY INDUCTION) CAN BE GROWN FROM uw, IE $uw\ldots$ IS GR. OF G'

APPARENTLY $v_n\, u\, w\ldots$ IS GROWING OF G. □

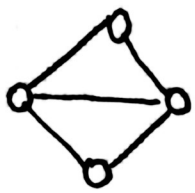
THEOREM: $|V_G| \geq 4$ AND G IS EDGE-MAXIMAL WITH NO K_4 MINOR $\Rightarrow$ G IS A 2-TREE.

PROOF:

[EXERCISE: A 2-TREE WITH ≥ 4 VERTICES IS EDGE MAXIMAL K_4-MINOR FREE]

INDUCTION

1) $|V(G)| = 4 \Rightarrow K_4 \setminus e$

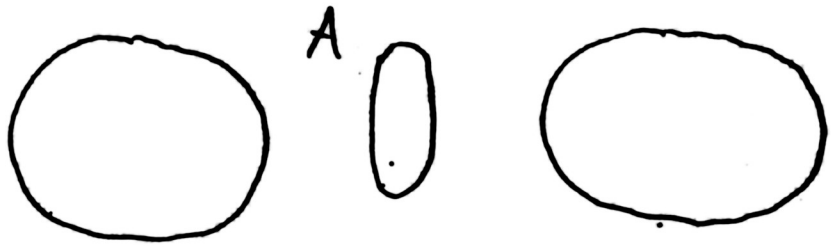


THIS IS A 2-TREE

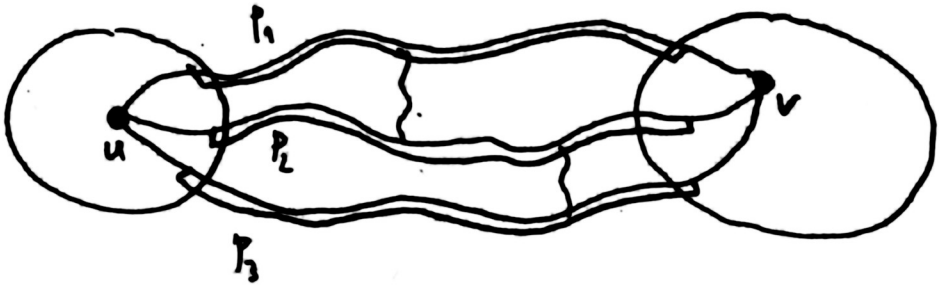
2) $|V(G)| > 4$.

G IS NOT A COMPLETE GRAPH $\Rightarrow k_v(G) = k$

$\exists A \subseteq V_G$ $|A| = k$ A VERTEX CUT



a) $|A| \geq 3$



TAKE u, v IN DIFFERENT COMPONENTS.

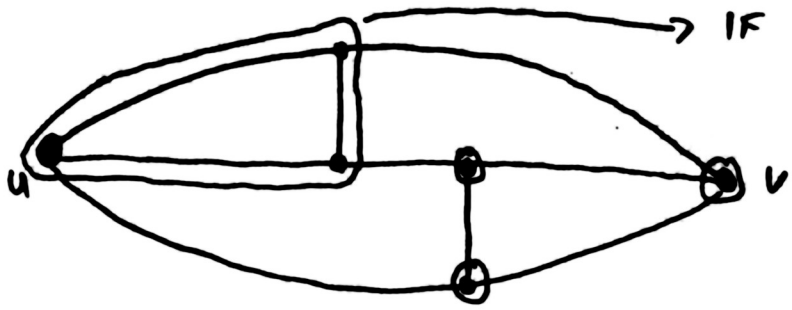
THERE EXIST 3 PATHS JOINING u AND v

$G \setminus \{u, v\}$ IS CONNECTED

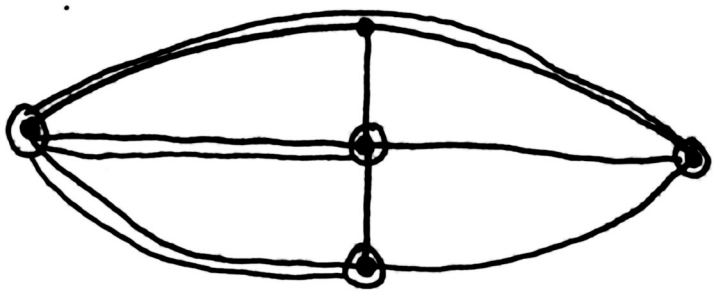
THERE IS A WAY HOW TO CONNECT AT LEAST TWO PAIRS OF PATHS P_1, P_2, P_3.

FOR INSTANCE $P_1 - P_2$, $P_2 - P_3$

THERE ARE TWO POSSIBILITIES



IF WE CONTRACT THIS PART, WE GET $K_4 <_m G'$.



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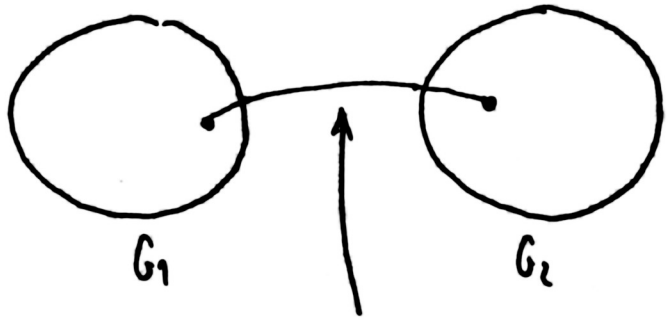
b) $|A| < 3$.

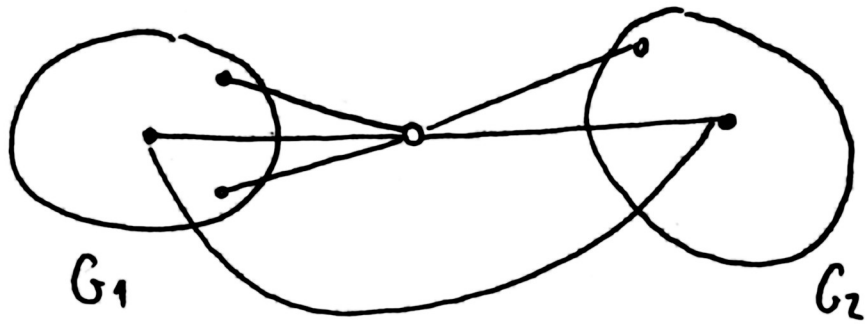
$|A| = 0$ .. G IS A DISCONNECTED GRAPH



WE CAN ADD AN EDGE, SO

G IS NOT EDGE-MAXIMAL

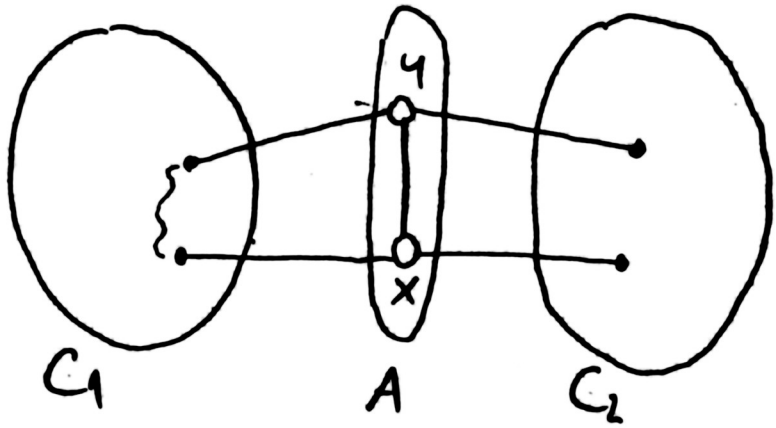
$|A| = 1$



WE CAN ADD AN EDGE
AND WE DO NOT CREATE A SUBDIV. OF K_4.
THE ONLY POSSIBILITY IS THAT K_4
LIVES IN G_1 OR G_2.

AGAIN G IS NOT EDGE-MAXIMAL

$|A| = 2$



A IS A MINIMAL VERTEX CUT.

WE ASSUME THAT G IS EDGE-MAXIMAL

$\Rightarrow$ VERTICES x, y IN A ARE CONNECTED.

(BECAUSE OTHERWISE WE COULD ADD THIS EDGE)

LET $G_1 = C_1 + A$, $G_2 = C_2 + A$

G_1 AND G_2 ARE EDGE-MAXIMAL WITH NO K_4 MINOR. (APPARENTLY TRUE)

BY INDUCTION G_1 AND G_2 ARE 2-TREES AND G_1 CAN BE GROWN ARBITRARILY, G_2 WILL BE GROWN FROM xy.

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DM1 012 KOMB. A GRAFY II

PROOF OF $\neg K_4 <_m G \Rightarrow \chi(G) \leq 3$.

IF G IS A GRAPH SUCH THAT $\neg K_4 <_m G$, WE CAN ADD EDGES TO G IN SUCH A WAY, THAT RESULTING G' IS EDGE-MAXIMAL SUPERGRAPH OF G WITH NO K_4 MINOR.

WE HAVE PROVEN THAT G' IS A 2-TREE.

IT IS EASY TO SEE THAT EVERY 2-TREE CAN BE COLOURED WITH 3 COLOURS.

$\Rightarrow \chi(G') \leq 3 \Rightarrow \chi(G) \leq 3$. $\square$

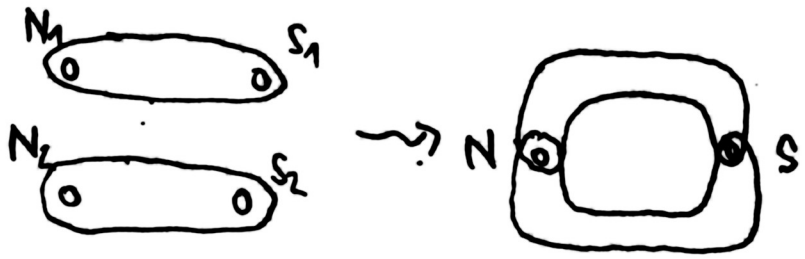
EXERCISE: KURATOWSKI TYPE THEOREM FOR OUTERPLANAR GRAPHS

EXERCISE: TREE-WIDTH, SERIES PARALLEL GRAPHS.

SERIES - PARALLEL GRAPHS

SERIES:

PARALLEL:



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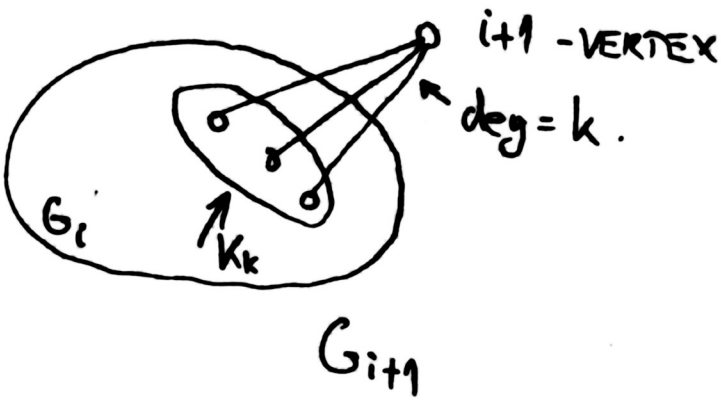
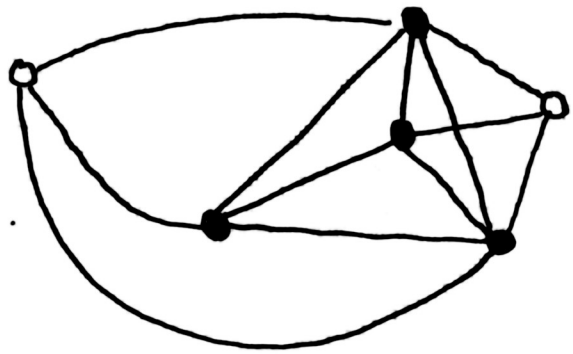
EX: DM1012 KOMB. A GRAFY II

DEF.: PARTIAL 2-TREE $\Leftrightarrow$ SUBGRAPH OF A 2-TREE.

DEF.: $k \in \mathbb{N}$ G IS A k-TREE $\Leftrightarrow$ G CAN BE CONSTRUCTED AS FOLLOWS: WE START FROM $G_{k+1} := K_{k+1}$



EXAMPLE: 3-TREE:



DEF.: PARTIAL k-TREE IS A SUBGRAPH OF k-TREE.

DEF.: TREEWIDTH OF G $\quad tw(G) :=$
$\min \{ k \mid G$ IS A PARTIAL k-TREE $\}$

$G = (V, E) \quad |V_G| = n \Rightarrow tw(G) \leq n-1.$

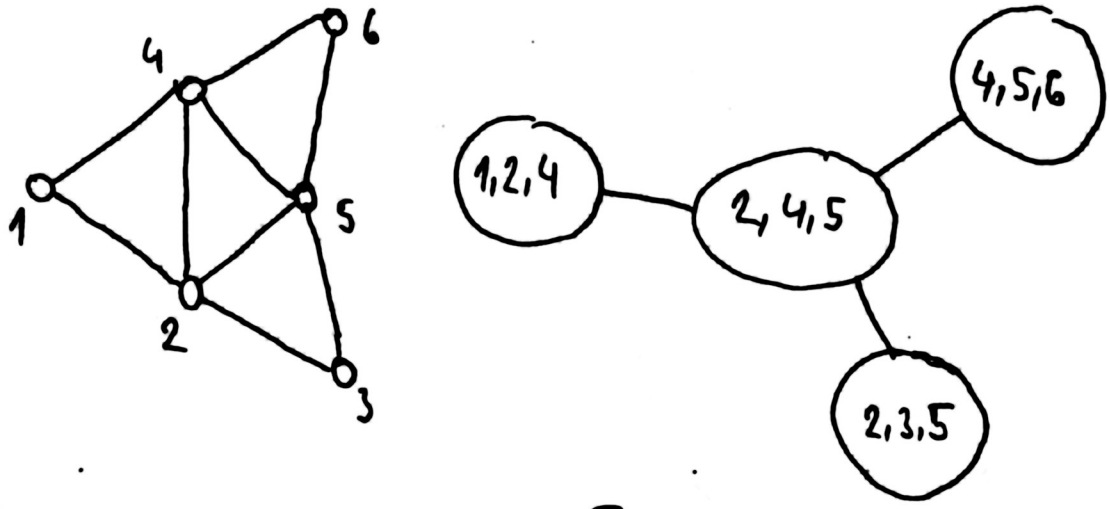
DEF.: TREE DECOMPOSITION OF $G = (V, E)$
IS A TREE $T = (B, F)$

$b \in B$

"BAGS" $b \subseteq V$

1) $\forall v \in V_G : \{ b \in B \mid v \in b \}$ INDUCES
A SUBTREE OF T

2) $\forall u, v \in V_G \quad \{u, v\} \in E \quad \exists b \in B : u, v \in b.$



TREE DECOMPOSITION T OF G HAS WIDTH $\underline{\underline{k+1}}$
$\Leftrightarrow \max \{ |b| \mid b \in B \} = k.$

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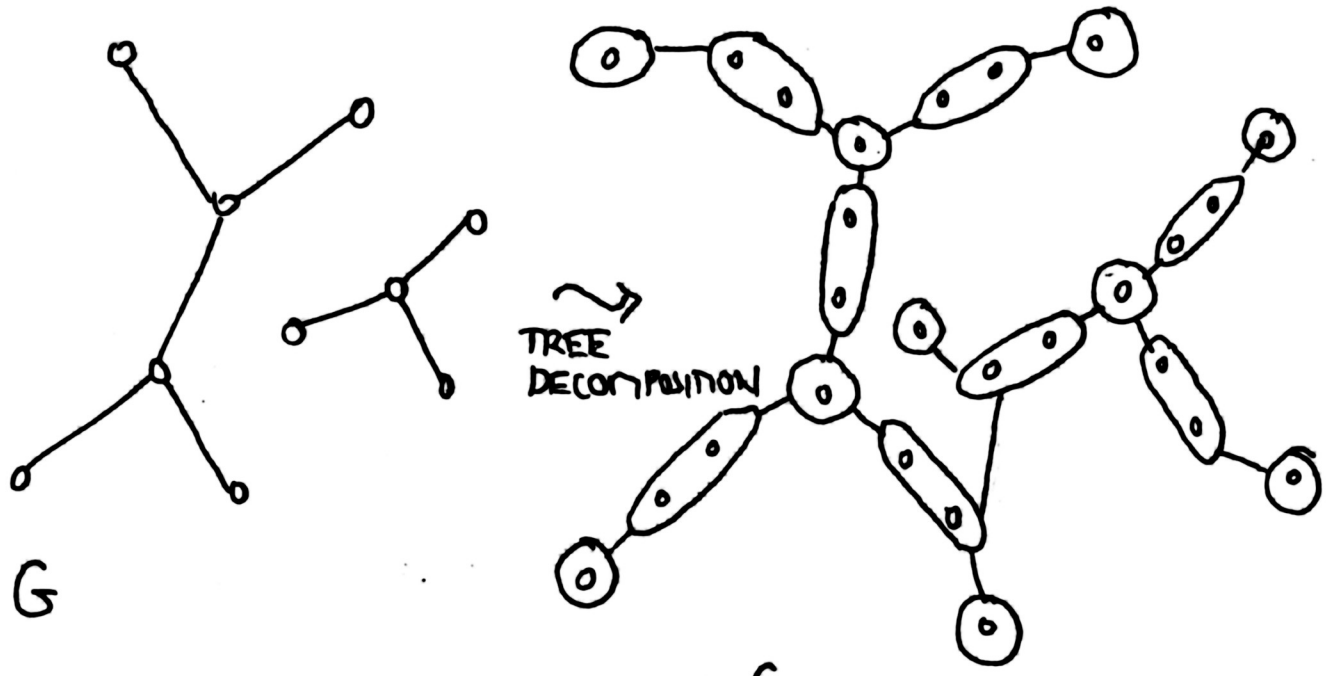
EX: DM1012 KOMB. A GRAFY II

DEF: $tw(G) := \min \{ width(T) \mid T$ IS A TREE DECOMPOSITION OF $G \}$

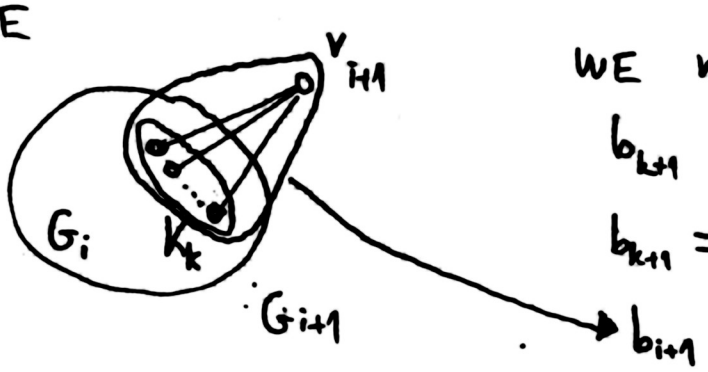


$tw(G) = 1$

1) G IS A k-TREE $\Rightarrow$ G HAS A TREE-DECOMPOSITION OF WIDTH $\leq k$.

FIRST IDEA: G IS A k-TREE $\Rightarrow$ G IS CONSTRUCTED FROM $G_{k+1} := K_{k+1}, \dots, G_n = G$

WHERE

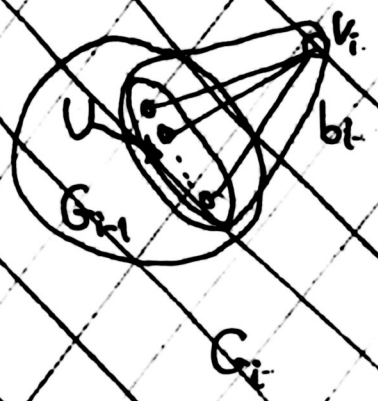


WE WILL HAVE BAGS $b_{k+1}, \dots, b_n$

$b_{k+1} = V(G_{k+1})$

APPARENTLY $|b_i| = k+1$ AND $G_i[b_i] = K_{k+1}$.

NOW WE HAVE TO DEFINE EDGES OF T:



b_i IS CONNECTED TO b_j, WHERE

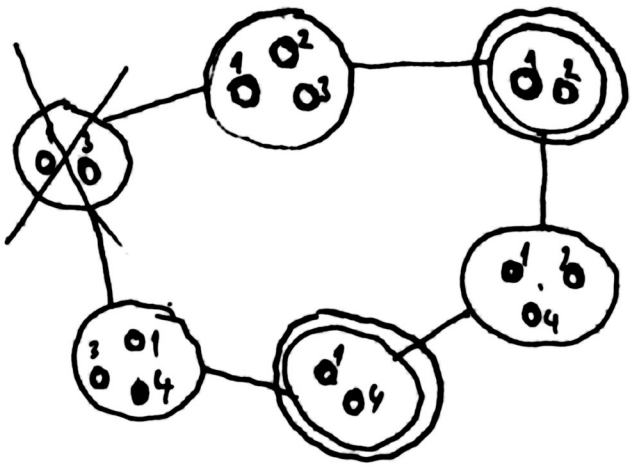
$$j = \max\{r \mid r \in U\}$$

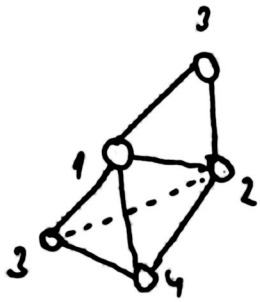
WE CAN SHOW BY INDUCTION THAT $\forall v \in V_G$ $\{b \in B \mid v \in b\}$ INDUCES A SUBTREE OF T

FOR $i = k+1$ WE HAVE JUST ONE BAG b_{k+1}

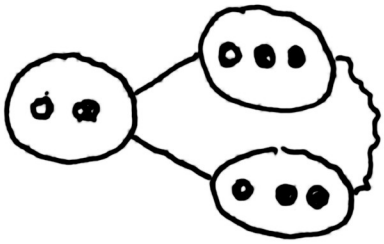
FOR $i > k+1$

⊗ THIS DOES NOT WORK FOR $k=1$





K_4 IS NOT ALLOWED IN A 2-TREE !!



$tw(K_{m,n}) = \min\{m,n\}$

$tw(K_n) = n-1$

$tw(C_n) = 2$

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EX: DM1012 KOMB. A GRAFY II

PROBLEM: $\forall$ PARTIAL k-TREE HAS A VERTEX OF DEGREE $\leq k$.

PROOF: LET $v_1, v_2, \ldots, v_n$ BE A GROWING OF A k-TREE, G IS A SUBGRAPH OF THIS k-TREE. WE CAN TAKE VERTEX v_i IN G WITH LARGEST INDEX $\Rightarrow$ THIS VERTEX HAS A DEGREE $\leq k$.

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DM1012 KOMB. A GRAFY II

DEF.: BIPARTITE GRAPH G WITH PARTS X, Y
THE DENSITY OF $G = \frac{|E_G|}{|X|.|Y|} = d(G) = d(X,Y)$.

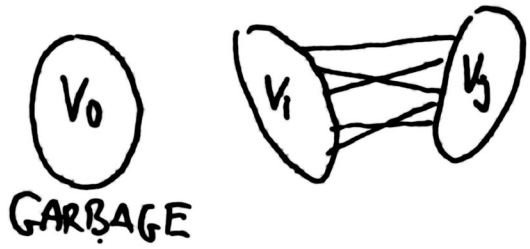
$d(X,Y) \in \langle 0; 1 \rangle$

BIPARTITE GRAPH G WITH PARTS X, Y IS ε-REGULAR $0<\varepsilon<1$
IF: $\forall X' \subseteq X$, $\forall Y' \subseteq Y$ SUCH THAT

$|X|' \geq \varepsilon |X|$, $|Y'| \geq \varepsilon |Y|$

$|d(X',Y') - d(X,Y)| \leq \varepsilon$ HOLDS.

DEF.: ε-REGULAR PARTITION OF G INTO k PARTS
IS A PARTITION OF $V(G)$ INTO SETS $V_0, V_1, \dots, V_k$
$|V_0| \leq \varepsilon . |V(G)|$ ... V_0 IS GARBAGE,
$|V_1| = \dots = |V_k|$
ALL BUT εk^2 PAIRS $1 \leq i < j \leq k$ FORM ε-REGULAR PAIR,
I.E. THE BIPARTITE SUBGRAPH OF G "INDUCED" (*)
BY PARTS V_i AND V_j IS ε-REGULAR.



(*) $G[V_i \cup V_j]$ WITHOUT EDGES INSIDE V_i AND V_j.

THEOREM (SZEMÉREDI REG. LEMMA):
$\forall \varepsilon > 0, m, n_0 \in \mathbb{N}\ \exists M\ \forall G\ |V_G| \geq n_0\ \exists k\ m \leq k \leq M$
G HAS AN ε-REGULAR PARTITION INTO k PARTS.

EVERY GRAPH G HAS ε-REGULAR PARTITION INTO 1 PART AND ALSO INTO $|V_G|$ PARTS.

LEMMA: LET H BE ε-REGULAR BIPARTITE GRAPH WITH PARTS A AND B AND WITH DENSITY d, THEN AT LEAST $(1-\varepsilon)|A|$ VERTICES OF A HAVE DEGREE AT LEAST $(d-\varepsilon).|B|$.

"EXPECTED" DEGREE OF A VERTEX OF $A \approx d.|B|$.

PROOF: ASSUME THE OPOSITE. LET A' BE THE SET OF VERTICES OF A WITH DEGREE $< (d-\varepsilon).|B|$. BY THE ASSUMPTION, $|A'| > \varepsilon.|A|$. APPLY THE CONSTRAINT FROM THE DEFINITION FOR SETS A', $B' = B$.

$\#$EDGES BETWEEN A' AND $B' = B$ $\geq (d-\varepsilon).|A'|.|B|$.

$\left[\left| \frac{\#\text{EDGES BETWEEN } A' \text{ AND } B'}{|A'|.|B'|} - d \right| \leq \varepsilon, \quad d-\varepsilon \leq \frac{\#\text{EDGES}}{|A'|.|B'|} \right]$

AND ALSO $\#$EDGES $< (d-\varepsilon).|B|.|A'|$, CONTRADICTION.

$\forall \varepsilon_0 > 0 \; \exists n_0, \delta \; \forall G \; |V_G| \geq n_0$, ONE OF THE FOLLOWING HOLDS (NOT EXCLUSIVELY):

(i) G HAS $\geq \delta \cdot n^3$ TRIANGLES

(ii) G CAN BE MADE Δ-FREE BY REMOVING $\leq \varepsilon_0 n^2$ EDGES

PROOF: $\varepsilon := \frac{\varepsilon_0}{10}$, $M := \max\{3, \frac{1}{\varepsilon}\} \Rightarrow \exists n_0, M$ (REGULARITY LEM.)

LET $d_0 := \frac{\varepsilon_0}{2}$

FROM REGULARITY LEMMA WE GET PARTITION $\mathcal{P}$ OF V_G INTO $V_0, \dots, V_k$, WHERE $m \leq k \leq M$

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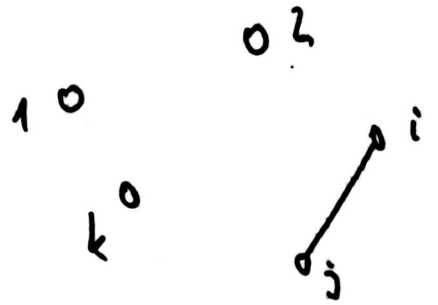
DM1012 KOMB. A GRAFY II

REGULARITY GRAPH OF G WITH RESPECT TO A THRESHOLD DENSITY d



ij IS AN EDGE

(i) V_i AND V_j IS AN ε-REGULAR PAIR

(ii) $d(V_i, V_j) \geq d_0$

(1) ASSUME THAT THE REG. GRAPH HAS NO Δ

$\Rightarrow$ REMOVE ALL EDGES EXCEPT FOR EDGES BETWEEN V_i AND V_j S.T. ij IS AN EDGE OF REG. GRAPH.

APPARENTLY THE RESULTING GRAPH IS Δ-FREE.

THE NUMBER OF EDGES REMOVED:

(i) INCIDENT WITH VERTICES OF V_0 ..

.. AT MOST $(\varepsilon \cdot n) \cdot n \leq \frac{\varepsilon_0}{10} \cdot n^2$

(ii) EDGES INSIDE V_i: $\underbrace{\leq k\left(\frac{n}{k}\right)^2}_{\text{BECAUSE } |V_i| \leq \frac{n}{k}} = \frac{n^2}{k} \leq \frac{n^2}{m} \leq \varepsilon \cdot n^2 =$

$= \frac{\varepsilon_0}{10} \cdot n^2$

(iii) EDGES BETWEEN V_i AND V_j, V_i AND V_j IS NOT ε-REGULAR PAIR

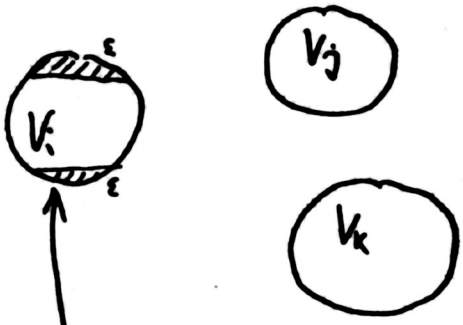
$\leq \underbrace{\varepsilon \cdot k^2}_{\text{NUMBER OF SUCH PAIRS } (V_i, V_j)} \left(\frac{n}{k}\right)^2$

(iv) EDGES BETWEEN SPARSE V_i, V_j

$$\underbrace{\leq k^2}_{\text{ALL PAIRS } (V_i, V_j)} \cdot d_0 \left(\frac{n}{k}\right)^2 = d_0 n^2 = \frac{\varepsilon_0}{2} n^2$$

TOGETHER AT MOST $\varepsilon_0 \cdot n^2$ EDGES.

(2) SUPPOSE THERE AN Δ IN <u>REGULARITY GRAPH</u>



V_i HAS AT LEAST $\geq (1-\varepsilon)|V_i|$ VERTICES

WITH $\geq (d_0 - \varepsilon)$ NEIGHBOURS IN V_j

$\geq (d(V_i, V_j) - \varepsilon)|V_j|$

V_i HAS AT LEAST $(1-\varepsilon) \cdot |V_i|$ VERTICES

WITH $\geq (d_0 - \varepsilon) \cdot |V_k|$ NEIGHBOURS IN V_k.

$\Rightarrow$ V_i HAS AT LEAST $(1-2\varepsilon) \cdot |V_i|$ VERTICES

WITH $\geq (d_0 - \varepsilon)|V_j|$ NEIGHBOURS IN V_j

$\geq (d_0 - \varepsilon)|V_k|$ NEIGHBOURS IN V_k.

$d_0 - \varepsilon = 5\varepsilon - \varepsilon = 4\varepsilon \geq \varepsilon$.

CHOOSE SUCH VERTEX v AND LET

N_j BE THE NEIGHBOURS IN V_j

N_k —"— IN V_k

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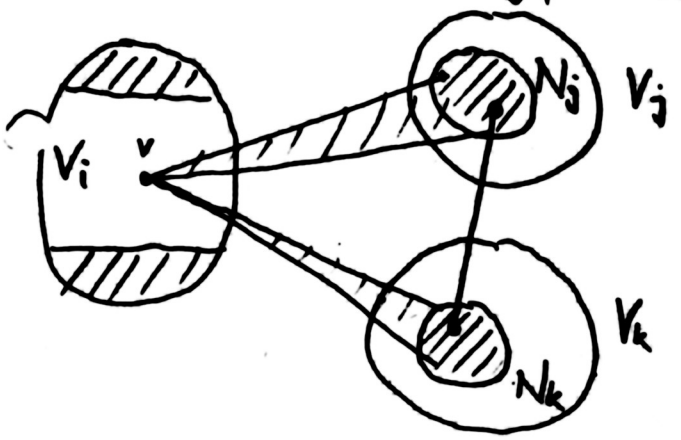
DM10 12 KOMBINATORIKA A GRAFY II

$|N_j| \geq \varepsilon |V_j|, \quad |N_k| \geq \varepsilon |V_k|$

THERE ARE AT LEAST $(d(V_j, V_k) - \varepsilon)|N_j|\,|N_k|$ EDGES BETWEEN N_j AND N_k.

$\geq (d_0 - \varepsilon)\,|N_j| \cdot |N_k| \geq (d_0 - \varepsilon)^3 \cdot |V_j| \cdot |V_k|$



$\#\Delta \geq$

$(1 - 2\varepsilon) \cdot (d_0 - \varepsilon)^3 \cdot$

$|V_i| \cdot |V_j| \cdot |V_k| \geq$

$\geq \frac{8}{10} \left(\frac{4}{10}\right)^3 \varepsilon_0^3 \left(\frac{9}{10k}\right)^3 \cdot n^3 \geq$

$\geq \frac{2^3 \cdot 3^6}{10^7} \cdot \varepsilon_0^3 \frac{1}{M^3}\, n^3$ ■

$\delta :=$

$\frac{(1-\varepsilon)n}{k} \leq |V_i| \leq \frac{n}{k}$

GARBAGE IN V_0

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EX: DMI D12 KOMB. A GRAFY II

ε-REG. BIP. GR.
$\forall X' \subseteq X, Y' \subseteq Y$
$|X'| \geq \varepsilon|X|, |Y'| \geq \varepsilon|Y|$
$|d(X',Y') - d(X,Y)| \leq \varepsilon$

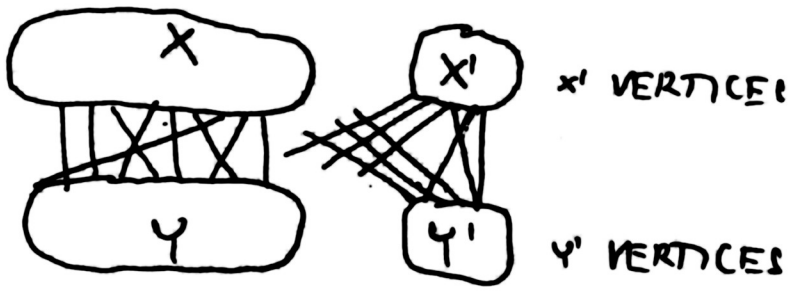
REGULARITY LEMMA: $\forall \varepsilon > 0, m \ \exists M \ \exists n_0$
$\forall G, \ |V(G)| \geq n_0 \ \exists k \ m \leq k \leq M$
G HAS ε-REGULAR PARTITION INTO k PARTS.

IT IS SUFFICIENT TO SET $n_0 := m$ INSTEAD OF $\exists n_0$

DEF.: ε-REGULAR* PARTITION OF G INTO k PARTS $V_1, \dots, V_k$: $\forall i,j \ \big| |V_i| - |V_j| \big| \leq 1$
EVERY BUT εk^2 PAIR FORMS ε-REGULAR BIPARTITE GRAPH...

REGULARITY LEMMA*: $\forall \varepsilon > 0 \ \exists M \ \forall G \ \exists k \leq M$ ...

L: (X,Y) $\forall \varepsilon > 0 \ \exists n_0 \ \exists \delta > 0$ ε-REGULAR BIPARTITE GRAPH $|X| = |Y| = n \geq n_0$



$|X' \cup Y'| \leq \delta n$

$\Rightarrow (X \cup X', Y \cup Y')$ IS 2ε-REGULAR BIP. GRAPH.

WE WANT TO PROVE THAT $\forall \mathcal{X} \subseteq X \cup X' \ \mathcal{Y} \subseteq Y \cup Y'$:
$|\mathcal{X}| \geq C_\varepsilon \cdot |X \cup X'|, \ |\mathcal{Y}| \geq C_\varepsilon |Y \cup Y'|$:
$|d(\mathcal{X}, \mathcal{Y}) - d(X \cup X', Y \cup Y')| \leq C_\varepsilon$

NOTE:

$d := d(X,Y)$

$\tilde{d} := d(X \cup X', Y \cup Y')$

$|d - \tilde{d}|$

$$\tilde{d} = \frac{\#e(X \cup X', Y \cup Y')}{|X \cup X'| \cdot |Y \cup Y'|}$$

$$\tilde{d} \leq \frac{\#e(X \cup X', Y \cup Y')}{|X| \cdot |Y|} \leq \frac{\#e(X,Y) + \#e(X', Y' \cup Y) + \#e(Y', X \cup X')}{|X| \cdot |Y|}$$

$$\leq d + \frac{\overset{\leq \delta n}{|X'|} \cdot |Y' \cup Y| + \overset{\leq \delta n}{|Y'|} \cdot |X \cup X'|}{|X| \cdot |Y|} \leq$$

$$\leq d + \frac{2\delta(1+\delta)n^2}{n^2}$$

$$\tilde{d} \geq \frac{\#e(X,Y)}{(1+\delta)|X| \, (1+\delta)|Y|} = \frac{1}{(1+\delta)^2} \cdot d$$

δ ZVOLÍME TAK, ŽE $|\tilde{d} - d| < \varepsilon$

PODOBNE POSTUPUJEME PRE $\mathcal{X}$ A $\mathcal{Y}$

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EX: DM1012 KOMB. A GRAFY II

(X,Y) ε-REGULAR BIPARTITE GRAPH

$d := d(X,Y) > \varepsilon$, $|X| = |Y| = n$

$$|\{(x,x') \in X \times X,\ |N(x) \cap N(x')| \leq (d-\varepsilon)^2 |Y|\}| \leq$$
$$\leq 2\varepsilon |X|^2.$$

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DMI012 KOMB. A GRAFY II

THE DENSITY OF A BIPARTITE GRAPH G WITH PARTS A AND B, DENOTED BY $d(A,B)$, IS $\frac{\#\text{EDGES OF } G}{|A|\cdot|B|}$.

A BIPARTITE GRAPH G IS ε-REGULAR IF
$\forall A' \subseteq A \; B' \subseteq B, \; |A'| \geq \varepsilon |A|, \; |B'| \geq \varepsilon |B|$:
$|d(A',B') - d(A,B)| \leq \varepsilon$.

AN ε-REGULAR PARTITION OF A GRAPH G INTO k PARTS
$V(G) = V_0 \cup V_1 \cup \dots \cup V_k, \quad |V_1| = \dots = |V_k|, \; |V_0| \leq \varepsilon \cdot |V_G|$.
ALL BUT εk^2 PAIRS V_i AND V_j ARE ε-REGULAR, IE., THE BIPARTITE GRAPH FORMED BY THE $V_i - V_j$ EDGES IS ε-REGULAR.

SRL: $\forall \varepsilon > 0, m \; \exists n_0, M$ SUCH THAT EVERY GRAPH $\geq n_0$ VERTICES HAS AN ε-REGULAR PARTITION INTO k PARTS, WHERE $m \leq k \leq M$.

LEMMA 1: LET G BE AN ε-REGULAR BIPARTITE GRAPH WITH PARTS A AND B. LET $Y \subseteq B$ AND $|Y| \geq \varepsilon |A| \cdot |B|$. AT LEAST $(1-\varepsilon) \cdot |A|$ VERTICES OF A HAVE $\geq (d-\varepsilon)|Y|$ NEIGHBOURS IN Y.

PROOF: LET A' BE THE VERTICES OF A WITH $< (d-\varepsilon) \cdot |Y|$ NEIGHBOURS IN Y.
IF $|A'| \leq \varepsilon |A|$, THEN OK.
IF $|A'| > \varepsilon |A|$, THEN EXAMINE THE DENSITY $d(A',Y)$.
$\#$EDGES BETWEEN A' AND Y $< |A'| \cdot (d-\varepsilon)|Y|$
$\Rightarrow d(A',Y) < d - \varepsilon$, WHICH CONTRADICTS ε-REGULARITY. $\square$

WHAT IS THE MAXIMUM NUMBER OF EDGES OF AN n-VERTEX GRAPH WITHOUT A SUBGRAPH ISOMORPHIC TO A FIXED GRAPH H.

THEOREM (TURÁN): LET $ex(n, H)$ BE THE MAXIMUM NUMBER OF EDGES OF AN H-FREE GRAPH ON n VERTICES.

$$ex(n, K_r) = t_{r-1}(n)$$

THE # EDGES OF A COMPLETE $(r-1)$-PARITE GRAPH WITH PARTS OF SIZES $\lfloor \frac{n}{r} \rfloor$ AND $\lceil \frac{n}{r} \rceil$.

EXERCISE $\lim_{n\to\infty} \frac{t_{r-1}(n)}{n^2/2} = \frac{r-2}{r-1}$

EDGES IN THE EXTREMAL GRAPH

$$\approx \binom{n}{2}\frac{r-2}{r-1} + O(n)$$

THEOREM (ERDÖS - STONE):

$$\lim_{n\to\infty} \frac{ex(n, H)}{\binom{n}{2}} = \lim_{n\to\infty} \frac{ex(n, H)}{n^2/2} = \frac{\chi(H)-2}{\chi(H)-1}.$$

$$\lim_{n\to\infty} \frac{ex(n, H)}{n^2/2} \geq \frac{\chi(H)-2}{\chi(H)-1}$$

WHERE $r = \chi(H)$

TAKE THE TURAN GRAPH WITH $t_{r-1}(n)$ EDGES.

SINCE THE CHROMATIC NUMBER OF THIS GRAPH IS AT MOST $\leq r-1$ $\Rightarrow$ THIS GRAPH IS H-FREE

$\Rightarrow ex(n, H) \geq t_{r-1}(n)$.

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DM1012 KOMB. A GRAFY II

THE s-CLONE $G^{(s)}$ OF A GRAPH G ...

$V(G^{(s)}) = V(G) \times \{1, \dots, s\}$

$[u,i] \sim [v,i] \in E(G^{(s)}) \Leftrightarrow uv \in E(G)$

THE TURAN GRAPH WITH $r-1$ PARTS AND $(r-1)s$ VERTICES IS THE s-CLONE OF K_{r-1}.

FOR AN ε-REGULAR PARTITION WITH k-PARTS, THE REGULARITY GRAPH R WITH THRESHOLD d_0 WAS A GRAPH WITH VERTICES $1, \dots, k$ AND $i \sim j \Leftrightarrow$:

(i) V_i AND V_j IS ε-REGULAR

(ii) $d(V_i, V_j) \geq d_0$

<u>LEMMA</u>: $\forall d_0, s, \Delta \; \exists \varepsilon_0 > 0 \; \forall G$ WITH ε_0-REGULAR PARTITION, H WITH MAX DEGREE $\leq \Delta$, $\forall |V_i| \geq \frac{2s}{d_0} \Delta$

$H \subseteq R^{(s)} \Rightarrow H \subseteq G$ WHERE R IS REGULARITY GRAPH WITH THRESHOLD d_0.

PROOF: CHOOSE ε_0 SUCH THAT

$$(d_0 - \varepsilon_0)^{\Delta} - \Delta \varepsilon_0 \geq \frac{1}{2} d_0^{\Delta}.$$

THE VERTICES OF H ... $w_1, \dots, w_n$

$H \subseteq R^{(s)}$.

CANDIDATE SETS $Y_1, \dots, Y_n$

FIXING $w_1', \dots, w_n'$ ONE BY ONE

IN THE j-TH STEP: $w_1', \dots, w_j'$ ARE FIXED

$w_1', \dots, w_j', Y_{j+1}, \dots, Y_n$

$w_i \sim w_{i'}$ $\quad i, i' \leq j \Rightarrow w_i' \sim w_{i'}'$

$i < j < i' \Rightarrow w_i'$ IS COMPLETE TO $Y_{i'}$

$Y_i = V_a$ WHERE V_a IS THE PART OF THE PARTITION CORRESPONDING TO w_i

$w_i = [a, *]$

$\in \{1, \dots, s\}$

AT THE BEGINNING $|Y_i| = |V_a|$

IF δ NEIGHBOURS OF w_i ARE FIXED,

THEN $|Y_i| \geq (d_0 - \varepsilon)^{\delta} \cdot |V_a|$

$w_1', \dots, w_j'$ FIXED. I HAVE TO CHOOSE $w_{j+1}' \in Y_{j+1}$

WE DO NOT CARE ABOUT VERTICES $w_1', \dots, w_j'$.

ANY $w' \in Y_{j+1}$ IS CONNECTED TO ALL $w_1', \dots, w_j'$

ALL BUT $\varepsilon |V_a|$ VERTICES OF Y_{j+1} ARE ADJACENT TO AT LEAST $(d_0 - \varepsilon) |Y_{j+1}|$ VERTICES OF Y_{j+1}

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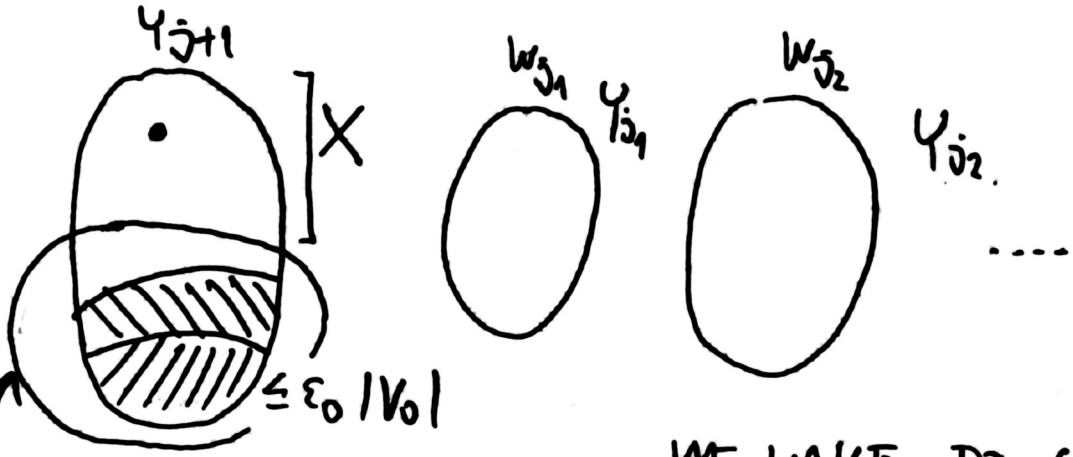
DM1012 KOMB. A GRAFY II



CHOOSE w'_{j+1}

$\le \Delta \varepsilon_0 |V_a|$

WE HAVE TO CARE ONLY TO NOT TO SHRINK THE SETS $Y_{j+2}, \dots, Y_n$ TOO MUCH

$$\Rightarrow \quad |X| \ge (d_0 - \varepsilon_0)^{\Delta} |V_a| - \Delta \varepsilon_0 |V_a| \ge$$
$$\ge \frac{1}{2} d_0^{\Delta} |V_a| \ge \underline{\underline{s}} .$$

WE CHOOSE w'_{j+1} TO BE DISTINCT FROM $w'_1, \dots, w'_j$

AND SHRINK Y_{j_1} TO $Y_{j_1} \cap N(w'_{j+1})$

$\boxed{\text{SEE THE PROOF IN THE GRAPH THEORY BOOK}}$

PROOF OF ERDÖS-STONE THEOREM :

"$\leq$" $H \subseteq K_r^{(s)}$ WHERE s IS # VERTICES OF H

$\forall \gamma > 0 \ \exists n_0$ SUCH THAT $\forall$ n-VERTEX GRAPH G

WITH $\geq t_{r-1}(n) + \gamma n^2$ EDGES $\Rightarrow K_r^{(s)} \subseteq G \Rightarrow H \subseteq G$.

$$\boxed{\Rightarrow \lim_{n\to\infty} \frac{ex(n,H)}{n^2/2} \leq \lim_{n\to\infty} \frac{t_{r-1}(n)}{n^2/2} + 2\gamma}$$

$d_0 = \gamma$, $\Delta = s\cdot(r-1) \xrightarrow{\text{LEMMA}} \varepsilon_0$,

WE CHOOSE $0 < \varepsilon \leq \varepsilon_0$ S.T.: $\delta := 2\gamma - \varepsilon^2 - 4\varepsilon - \varepsilon_0 - \frac{1}{m} > 0$

$m := \left\lceil \frac{1}{\gamma} + 1 \right\rceil$ $\qquad 2\gamma - \varepsilon_0 - \frac{1}{m} > 2\gamma - \gamma - \gamma = 0$

$\varepsilon, m \xrightarrow{\text{SRL}} M, n_0'$

$n_0 \overset{\text{def}}{:=} \max\left\{ n_0', \frac{2Ms}{d_0^{\Delta}(1-\varepsilon)}, \right.$ S.T. LIMIT

$\left. \lim_{n\to\infty} \frac{t_{r-1}(n)}{n^2/2} \right.$ IS WITHIN $\frac{\delta}{6}$ FROM $\left.\frac{r-2}{r-1}\right\}$

WE HAVE ε-REGULAR PARTITION, THE REGULARITY GRAPH WITH THRESHOLD d_0.

PARTS k IS BETWEEN m AND M.

$K \subseteq R \Rightarrow K_r^{(s)} \subseteq R^{(s)}$

$$|V_i| \geq \frac{n_0(1-\varepsilon)}{k} \geq \frac{n_0(1-\varepsilon)}{M} \geq \frac{2s}{d_0^{\Delta}}$$

$\Rightarrow K_r^{(s)} \subseteq G$

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DM10012 KOMB. A GRAFY II

IF $K_r \notin R$. LET $\ell := |V_i|$ $i = 1, \dots, k$

$$\#\text{EDGES OF } G \le \underbrace{\tfrac{1}{2}\varepsilon^2 n^2}_{\text{INSIDE } V_0} + \underbrace{\varepsilon n k \ell}_{V_0 \text{ VS REST}} + \underbrace{\varepsilon k^2 \ell^2}_{\text{IRREGULAR PAIRS}} +$$

$$+ \underbrace{\tfrac{1}{2} k^2 d_0 \ell^2}_{\text{SPARSE PAIRS}} + \underbrace{\tfrac{1}{2}\ell^2 k}_{\text{INSIDE } V_i} + \underbrace{\|R\| \cdot \ell^2}_{\#\text{EDGES OF } R}$$

n .. #VERTICES OF G $(k \cdot \ell \le n)$

$$\le \tfrac{1}{2}\varepsilon^2 n^2 + \varepsilon n^2 + \varepsilon n^2 + \tfrac{1}{2} d_0 n^2 + \tfrac{1}{2} n^2 / m + t_{r-1}(k) \cdot \ell^2$$

$$= \tfrac{1}{2} n^2 \underbrace{\left(\varepsilon^2 + 4\varepsilon + d_0 + \tfrac{1}{m}\right)}_{2\vartheta - \delta} + t_{r-1}(k) \cdot \ell^2$$

$$= \tfrac{1}{2} n^2 (2\vartheta - \delta) + t_{r-1}(k) \cdot \ell^2$$

IF m IS CHOSEN BIG ENOUGH

$$\le \vartheta n^2 - \frac{\delta n^2}{2} + \frac{r-2}{r-1} \frac{k^2}{2} \left(1 - \frac{\delta}{6}\right) \cdot \ell^2$$

$$\le \vartheta n^2 - \frac{\delta n^2}{2} + \frac{r-2}{r-1} \frac{n^2}{2} \left(1 - \frac{\delta}{6}\right)$$

$$\le \vartheta n^2 - \frac{\delta n^2}{2} + t_{r-1}(n) + \frac{\delta}{6} \frac{n^2}{2} < \underline{\underline{t_{r-1}(n) + \vartheta n^2}} \quad \blacksquare$$

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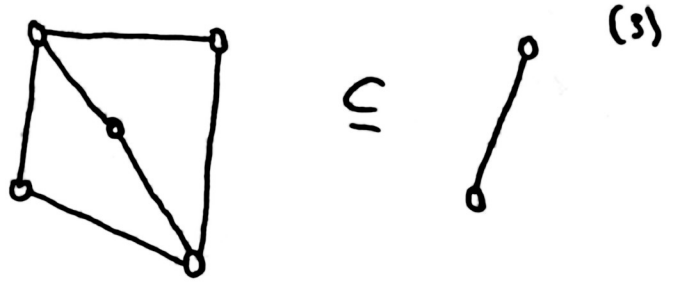
EX: DM1012 KOMB. A GRAFY II

LEMMA: $\forall s, \Delta, d_0 > 0 \ \exists \varepsilon_0 > 0 \ \forall G$ WITH
ε_0-REGULAR PARTITION $V_0, V_1, \dots, V_k$
$|V_i| \geq f(s, \Delta, d_0) \quad \forall i \in \{1, \dots, k\}$

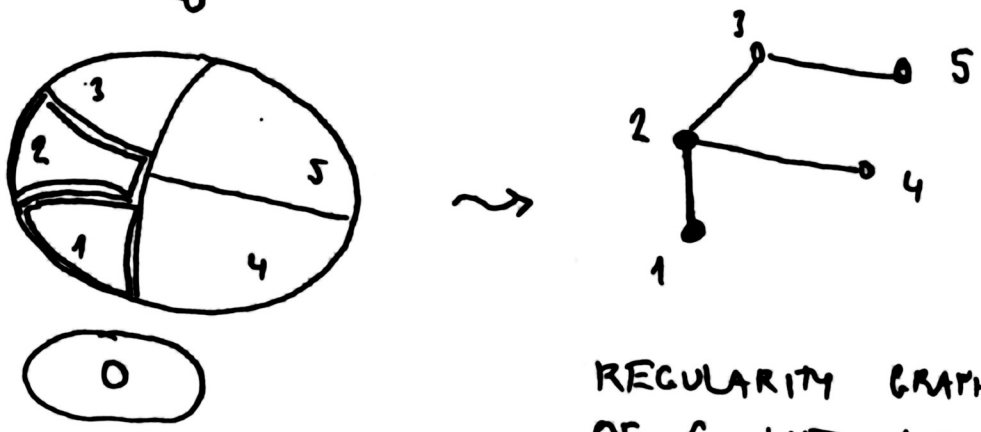
CLAIM: $\forall H \quad H \subseteq R^{(s)} \Rightarrow H \subseteq G$
$\Delta(H) \leq \Delta$

R IS THE REGULARITY GRAPH WITH DENSITY d_0.

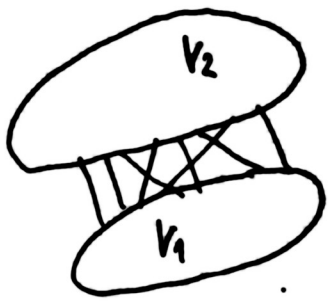
H: $K_{2,3}$



G:

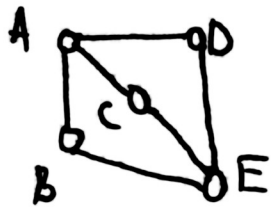


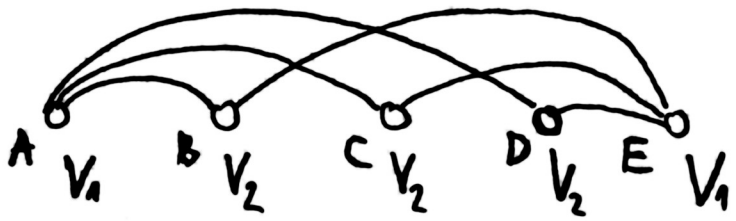
REGULARITY GRAPH R
OF G WITH DENSITY d_0

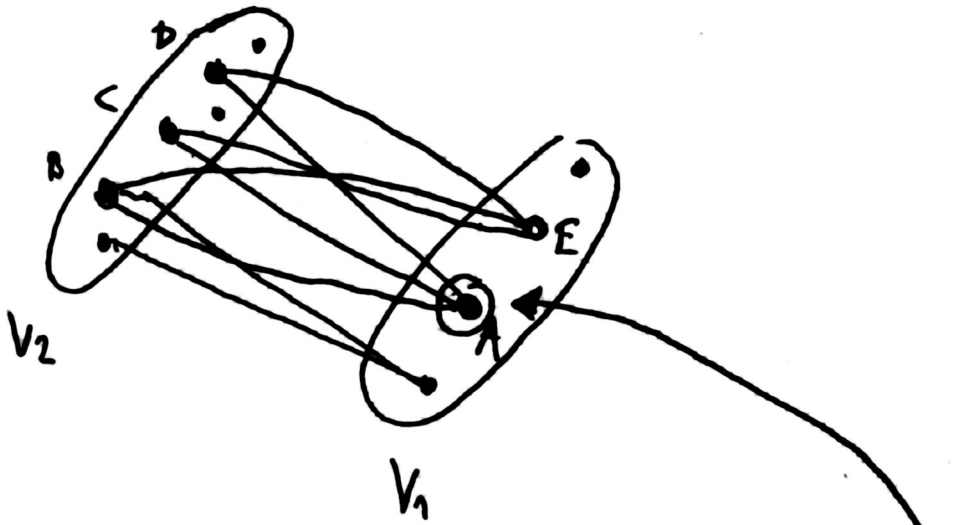


OUR AIM: FIND $K_{2,3}$
IN $V_1 \cup V_2$

H:







FIRST WE HAVE TO CHOOSE A IN V_1, THAT HAS AT LEAST $(d_0 - \varepsilon_0)|V_2|$ NEIGHBOURS IN V_2.

$\boxed{\text{L: AT MOST } \varepsilon_0 |V_1| \text{ VERTICES IN } V_1 \text{ HAVE LESS THAN } (d_0 - \varepsilon_0)|V_2| \text{ NEIGHBOURS IN } V_2.}$

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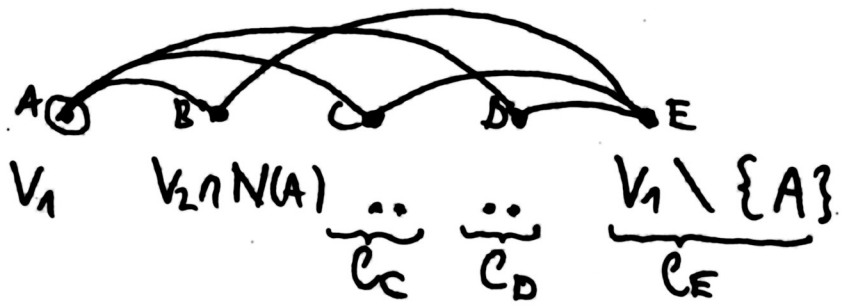
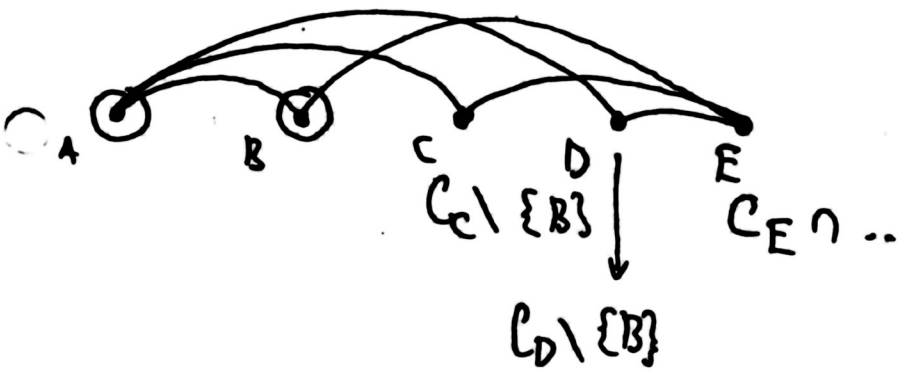
EX: DM1012 KOMB. A GRAFY II

NEXT WE HAVE TO CHOOSE B NOT IN V_2, BUT IN $\underline{V_2 \cap N(A)}$



WE HAVE TO CHOOSE B WITH ENOUGH NEIGHBOURS IN $V_1 \setminus \{A\}$:

AT LEAST $(d_0 - \varepsilon_0)|C_E|$ NEIGHBOURS IN C_E.



⋮

WE HAVE TO MAKE SURE THAT $(d_0 - \varepsilon_0)^{\Delta} |V_1| \geq \varepsilon |V_1|$.

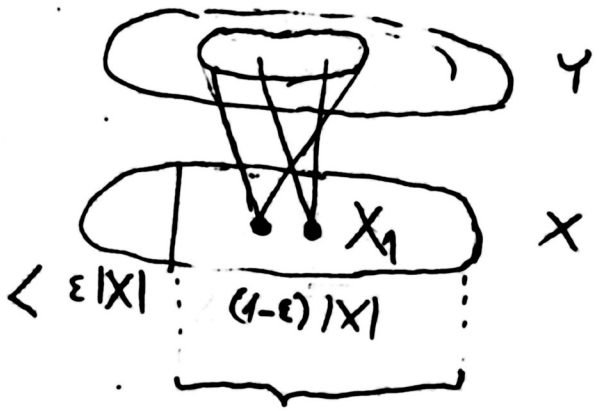
1) $\forall d > 0 \quad \forall \varepsilon > 0,\ \varepsilon < \frac{d}{2} \quad \exists n_0$

$\forall$ ε-REGULAR BIPARTITE GRAPH WITH PARTS X, Y SUCH THAT $|X| = |Y| = n \geq n_0$

THEN ALL PAIRS (x, x'), $x, x' \in X$ EXCEPT AT MOST $2\varepsilon |X|^2$ SATISFY THAT $|N(x) \cap N(x')| \geq (d-\varepsilon)^2 |Y|$.

BIPARTITE GRAPH G IS ε-REGULAR $\Leftrightarrow$
$\forall X' \subseteq X \quad Y' \subseteq Y \quad |X'| \geq \varepsilon |X|, |Y'| \geq \varepsilon |Y|$
$|d(X', Y') - d(X, Y)| < \varepsilon$.

IN X THERE IS AT MOST $\varepsilon |X|$ VERTICES x SUCH THAT $|N(x)| < (d-\varepsilon)|Y|$



IT SUFFICES TO PROVE $|N(X_1)| \geq \varepsilon |Y|$
$\forall x \in X_1 \ \deg x > (d-\varepsilon) \Rightarrow |N(x_1)|$

EXERCISE : PROVE THIS AT HOME !!!

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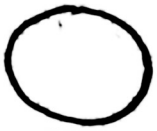
NDMI012 KOMBINATORIKA A GRAFY II

COLORING - GRAPHS ON SURFACES, INFINITE HYPERGRAPHS

(ORIENTABLE) SURFACE OF GENUS g IS OBTAINED FROM THE SPHERE BY ADDING g HANDLES.

$g = 0$

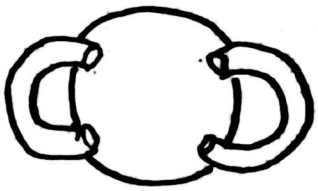


$g = 1$

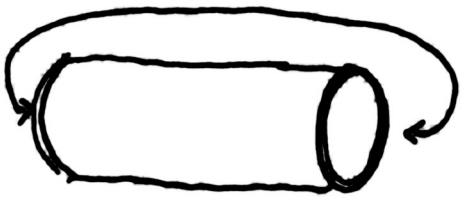
ANNULOID

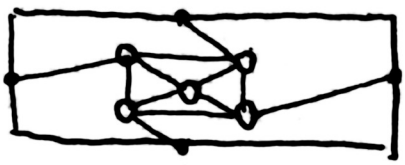
TORUS

$g = 2$



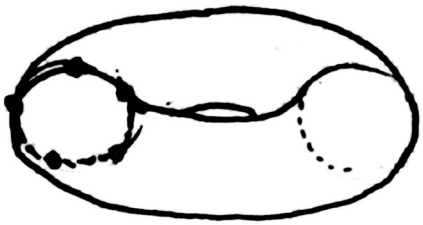
DEF: THE GENUS OF G $g(G) = \min g$ SUCH THAT G CAN BE DRAWN ON SURFACE OF GENUS g WITHOUT EDGE-CROSSING.





HOMEWORK: $g(K_6)$, $g(K_7)$, $g(K_8)$

DEF.: CELLULAR EMBEDING: EVERY FACE IS HOMEOMORPHIC TO A DISK.



THIS FACE IS NOT HOMEOMORPHIC TO A DISK

HOMEWORK: $\forall G$ EMBEDED ON S_g $\exists G' \supseteq G$ (G' IS A MULTIGRAPH) SUCH THAT G' HAS A CELLULAR EMBEDDING ON S_g, G IS SPANNING SUBGRAPH.

THEOREM (HEAWOOD):

$$H(g) = \max_{g(G)=g} \chi(G) = \left\lfloor \frac{1}{2}\left(7+\sqrt{1+48g}\right)\right\rfloor$$

(WE PROVE ONLY $\leq$)

EULER FORMULA FOR SURFACES

IF G IS CONNECTED AND HAS CELLULAR EMBEDING ON S_g

v = # OF VERTICES
e = # OF EDGES
f = # OF FACES

THEN

$$v+f = e+2(1-g)$$

PROOF: BY INDUCTION ON g

1) $g=0$: $v+f=e+2$

TRUE FOR EVERY CONNECTED PLANAR GRAPH G.

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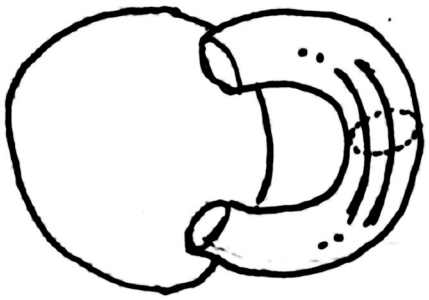
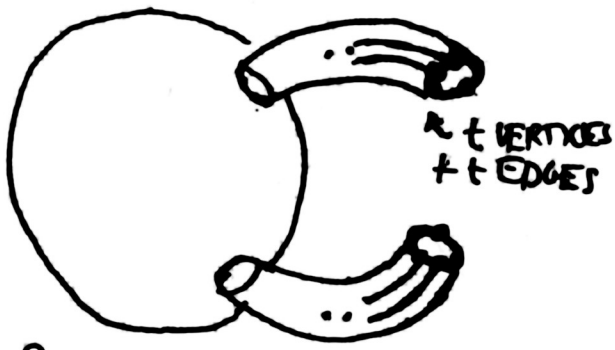
2) $g+1 \leftarrow g$



t = # OF EDGES ALONG THE HANDLE



G_{g+1}
v, e, f

G_g
$v' = v+2t$, $e' = e+t+2t$, $f' = f+t+2$

BY INDUCTION HYPOTHESIS: $v'+f' = e'+2(1-g)$

$\Rightarrow \quad v+f = e+2(1-(g+1))$ □

<u>COROLLARY</u>: FOR SIMPLE G SUCH THAT $g(G)=g$

$e \leq v+f+2(g-1)$.

(FOR G WITH CELLULAR EMBEDING "=" HOLDS
→ WE DELETE EDGES)

IF G IS SIMPLE WITH $g(G)=g$, THEN

$e \leq 3v+6(g-1)$ $\quad$ ($|E| \leq 3|V|-6$ FOR PLANAR G.)

PROOF: $3f \leq |\{(\text{edge}, \text{face}) \mid \text{edge IS ON THE BOUNDARY OF FACE}\}| \leq 2e$

EVERY face HAS AT LEAST 3 EDGES,
EVERY edge HAS AT MOST 2 FACES.

$\Rightarrow 3f \leq 2e$

$3e - 3v - 6(g-1) \leq 3f \leq 2e \Rightarrow \underline{\underline{e \leq 3v+6(g-1)}}$. □

PROOF OF "$\leq$" IN HEAWOOD (FOR $\mathcal{S} \geq 1$).

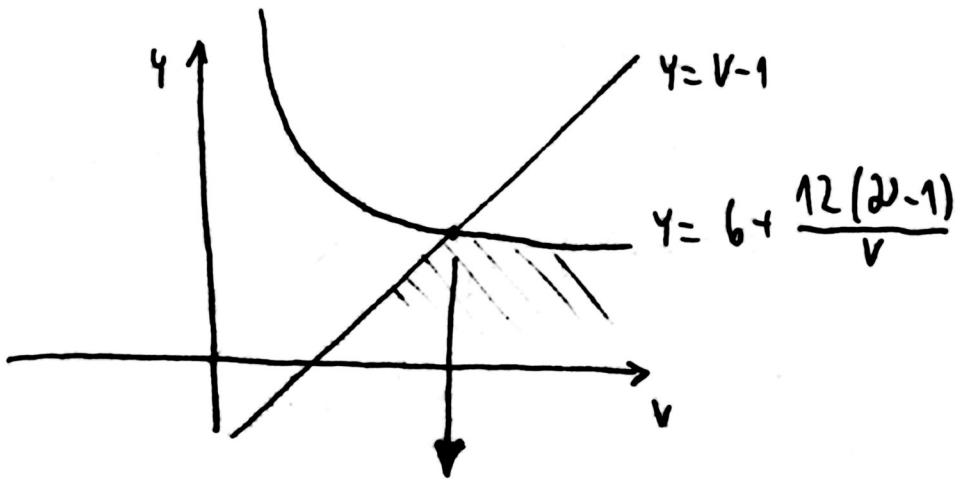
LET G BE SIMPLE EMBEDED ON $S\,\mathcal{S}$.

$\delta(G) \leq v-1$.

$$\sum_{w \in V_G} \deg_G w = 2e \leq 6v + 12(\mathcal{S}-1)$$

$$\Rightarrow \quad \delta(G) \leq \frac{6v+12(\mathcal{S}-1)}{v} = 6 + \frac{12(\mathcal{S}-1)}{v}$$

IE $\delta(G) \leq \min\left\{ v-1,\ 6+\frac{12(\mathcal{S}-1)}{v} \right\}$



$\boxed{\mathcal{S} \geq 1}$

$$v-1 = 6 + \frac{12(\mathcal{S}-1)}{v} \Rightarrow v_{1,2} = \frac{7 \pm \sqrt{49-48(1-\mathcal{S})}}{2}$$

$$\Rightarrow \delta(G) \leq \frac{7+\sqrt{1+48\mathcal{S}}}{2} - 1$$

WHAT DOES IT MEAN?

$$\delta(G) \leq H(\mathcal{S}) - 1$$

$\Downarrow$ INDUCTION (AS IN COLORING)

$\exists w \in V_G : \deg_G w \leq H(\mathcal{S}) - 1$

CONSIDER $G' = G - w$. BY INDUCTION

$\chi(G') \leq H(\mathcal{S})$

EXTEND SUCH A COLORING OF G' TO A COLORING G BY $H(\mathcal{S})$ COLORS.

$$\Rightarrow H(\mathcal{S}) \leq \frac{1}{2}\left(7 + \sqrt{1+48\mathcal{S}}\right). \quad \square$$

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NDMI012 KOMBINATORIKA A GRAFY II

$\nu = 0$ $\quad H(0) = 4$ HERE IT IS EASY TO PROVE LOWER BOUND. CONSIDER K_4

$\nu = 1$ $\quad H(1) = 7$

HOMEWORK: WHY OUR PROOF FAILS FOR $\nu = 0$?

HYPERGRAPH: $H = (V, E)$, $E \subseteq \mathcal{P}(V)$.

$\chi(H) = \min k$ SUCH THAT $\exists \varphi : V \to C$, $|C| = k$
$\forall e \in E : |\varphi(e)| \geq 2$.

THEOREM (COMPACTNESS): IF $H = (V, E)$ IS AN INFINITE HYPERGRAPH WITH <u>FINITE EDGES</u>. THEN FOR EVERY FINITE k

IF FOR EVERY FINITE SUBGRAPH H' OF H : $\chi(H') \leq k$
$\Rightarrow$ THEN $\chi(H) \leq k$.

PROOF FOR COUNTABLE H

$V = \{v_0, v_1, \dots\}$

DIAGONALIZATION

$V_i = \{v_0, \dots, v_i\}$ $\quad H_i$ $\quad \varphi_i : V_i \to \{1, \dots, k\}$ $\quad i = 0, 1, \dots$
COLOURING

1) WE LOOK WHAT COLOUR v_0 HAS INFINITELY MANY OF φ_i USE 1 COLOUR FOR v_0 (FOR INSTANCE red). WE SET $\varphi(v_0) = \text{red}$

NOW LOOK AT v_1.

AGAIN INFINITELY MANY OF φ_i USE 1 COLOUR FOR v_1. WE CONSIDER ONLY φ_i : $\varphi_i(v_0) = \text{red}$

$\vdots$

AFTER INFINITE NUMBER OF STEPS WE COLOUR THE WHOLE GRAPH

$\forall e \in E$ LET $i = \max\{j \mid v_j \in e\}$

THERE ARE INFINITELY MANY COLOURINGS φ_j S.T.:

$$\varphi_j(v_0) = \varphi(v_0), \ldots, \varphi_j(v_i) = \varphi(v_i)$$

$\Rightarrow$ IF WE TAKE ONE SUCH φ_j WE GET THAT $|\varphi(e)| \geq 2$. $\square$

<u>RAMSEY THEOREM</u> (RECALL)

$\forall k, n, p \ \exists N$ WE WRITE $N \to (n)^k_p$ SUCH THAT

$\forall X \ |X| = N \quad \forall \binom{X}{p} = A_1 \cup \ldots \cup A_k$

$\exists A \in \binom{X}{n} : \exists i : \binom{A}{p} \subseteq A_i$.

DEF: $H(N, n, p) = \left(\binom{N}{p}, \left\{ \binom{A}{p} \mid A \in \binom{N}{n} \right\} \right)$.

$\textcircled{\circ}$ $\forall k, n, p \ N : N \to (n)^k_p \Leftrightarrow \chi(H(N, n, p)) > k$.

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NDMI012 KOMBINATORIKA A GRAFY II

THEOREM: $\forall p, k : \omega \to (\omega)^k_p$.

PROOF (INDUCTION ON p).

a) $p=1$ $\quad$ o • o ● ● o ...

AT LEAST ONE COLOR MUST BE USED INFINITE MANY TIMES

b) $p+1 \leftarrow p$ $\quad$ (RECITATION)

INFINITE RAMSEY $\Rightarrow$ FINITE RAMSEY

$\forall p, k : \omega \to (\omega)^k_p$	$\forall p, n, k\ \exists N : N \to (n)^k_p$
$\forall p, k : \chi(H(\omega, \omega, p)) > k$	$\forall p, k, n\ \exists N : \chi(H(N, n, p)) > k$

LET US FIX p, k

$\Downarrow$

$\forall p, k, n : \chi(H(\omega, n, p)) > k$

$\Downarrow$ [COMPACTNESS]

$\exists$ FINITE $H' \leq H(\omega, n, p)$ SUCH THAT $\chi(H') > k$

$\Uparrow$

$\exists N : H' \leq H(N, n, p)$

$\chi(H') > k \Rightarrow \chi(H(N, n, p)) > k$

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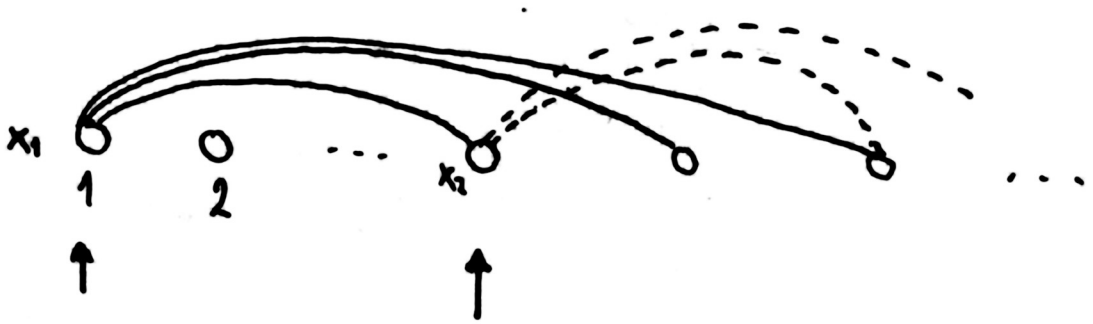
EX: NDMI012 KOMBINATORIKA A GRAFY II

$2 \leftarrow 1$: $\forall k \quad \omega \to (\omega)^k_2$

$\forall k$-COLOURING OF $E(K_\omega)$ $\exists$ INFINITE $\varphi \subseteq \omega$ SUCH THAT $K_\varphi \subseteq K_\omega$ HAS ALL EDGES OF THE SAME COLOUR.

PROOF: 1) $X_1 \subseteq \omega$ INFINITE, $x_1 \in X_1$ IS THE FIRST VERTEX, ALL THE EDGES FROM x_1 TO ANY LARGER VERTEX IN X_1 HAVE THE SAME COLOUR



LET $X_2 \subseteq X_1$ INFINITE, $x_2 \in X_2$ IS THE FIRST VERTEX, ALL EDGES FROM x_2 TO ANY LARGER VERTEX IN X_2 HAVE THE SAME COLOUR.

$\vdots$

SUPPOSE WE HAVE CONSTRUCTED A SEQUENCE $X_1 \supseteq X_2 \supseteq \dots$, $x_1 \in X_1$, $x_2 \in X_2$, ...

EVERY VERTEX x_i HAS ITS FAVOURITE COLOUR.

ONE COLOUR IS FAVOURITE FOR INFINITELY MANY OF THEM .. THIS IS OUR SET φ.

$p+1 \leftarrow p \qquad \forall k \quad \omega \to (\omega)^k_p$

1) WE WANT TO FIND INFINITE $X \subseteq \omega$, $v \in X$, ALL THE $(p+1)$-TUPLES FROM $\binom{X}{p+1}$ HAVE A COLOUR THAT ONLY DEPENDS ON THE SMALLEST ELEMENT OF THE TUPLE.

LET $c : \binom{\omega}{p+1} \to \{1, \dots k\}$ BE A COLOURING.

$x_1 := 1$, DEFINE COLOURING $c^{(1)} : \binom{\omega \setminus \{x_1\}}{p} \to \{1, \dots, k\}$ $\frown$
$c^{(1)}\left(\{y_1, \dots, y_p\}\right) := c\left(\{x_1, y_1, \dots, y_p\}\right)$.

BY INDUCTION HYPOTHESIS $\exists$ INFINITE SET $Y^{(1)} \subseteq \omega \setminus \{x_1\}$ SUCH THAT ALL p-TUPLES $\binom{Y^{(1)}}{p}$ HAVE THE SAME COLOUR c_1.

$\Rightarrow$ ALL $p+1$-TUPLES OF THE FORM $\{x_1, y_1, \dots, y_p\}$ FOR ALL $y_1, \dots, y_p \in Y^{(1)}$ HAVE $c(\{x_1, y_1, \dots, y_p\}) = c_1$.

ASSUME THAT WE HAVE: $x_1, \dots, x_j$, $Y^{(j)}$, SUCH THAT
$c(\{x_i, y_1, \dots, y_p\}) = c_i$ FOR ALL $y_1, \dots, y_p \in Y^{(j)}$ $\frown$

DEFINE $x_{j+1} := \min Y^{(j)}$, $x_{j+1} > x_j$.
AND $Y^{(j+1)} \subseteq Y^{(j)} \setminus \{x_{j+1}\}$ IS DEFINED ANALOGOUSLY.

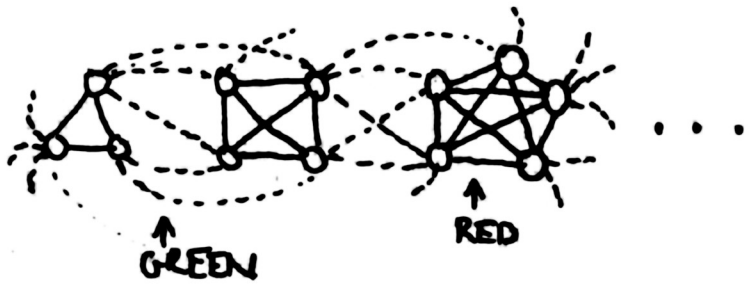
FOR EACH VERTEX x_i WE HAVE FAVOURITE COLOUR c_i.
SUPPOSE THAT A COLOUR c IS FAVOURITE FOR INFINITELY MANY VERTICES.
THEN $\varphi = \{x_i \mid c_i = c\}$ IS OUR DESIRED SET. $\square$

12.12.2007 Tu
PETER ČERNO

EX: NDMI012 KOMBINATORIKA A GRAFY II

1) TRUE OR FALSE:

IF I HAVE A COLOURING OF $E(K_\omega)$ BY RED AND GREEN SUCH THAT $\forall n \in \omega$ $\exists$ A RED COPY OF $K_n \subseteq K_\omega$, THEN $\exists$ INFINITE RED COMPLETE SUBGRAPH OF K_ω.



2) COUNTABLE DILWORTH: $\forall$ PARTIALLY ORDERED COUNTABLE SET $(P, \leq)$:

P CAN BE COVERED BY k CHAINS $\Leftrightarrow$ P HAS NO ANTICHAIN OF LENGTH $> k$.

3) $K_{X,Y}$ = COMPLETE BIPARTITE GRAPH WITH PARTS X, Y, X, Y COUNTABLY INFINITE.

IS IT TRUE THAT EACH 2-COLOURING OF $E(K_{X,Y})$:

✓ a) CONTAINS MONOCHROMATIC COPY OF $K_{n,n}$ $\forall n \in \mathbb{N}$?

✓ b) —"— $K_{n,\omega}$?

✗ c) —"— $K_{\omega,\omega}$?

08.01.2008 Čt
PETER ČERNO

DM012 KOMB. A GRAFY II

HALES - JEWETT THEOREM

A .. ALPHABET, FINITE SET $|A| = \ell$

SPECIAL SYMBOL $* \notin A$

"ROOT" $\rho \in (A \cup \{*\})^n$, CONTAINS AT LEAST ONE $*$

$\rho(a) :=$ WORD OBTAINED BY REPLACING EACH $*$ BY a.

"COMBINATORIAL LINE" ROOTED AT ρ : $\{\rho(a), a \in A\}$

$A = \{1, 2, 3\}$

A^2

11	12	13	... COMBINATORIAL LINE $1*$
21	22	23	... $2*$
31	32	33	... $3*$
$*1$		$**$	

THEOREM : $\forall k \exists N = N(k, \ell)$: EVERY k-COLORING OF A^N CONTAINS A MONOCHROMATIC COMBINATORIAL LINE.

REMARKS :

1) WLOG $A = \{1, 2, \dots, \ell\}$
2) $HJ(k, \ell) :=$ SMALLEST N SUCH THAT THE STATEMENT HOLDS
3) THIS IS AN IMPORTANT THEOREM

PROOF: INDUCTION ON ℓ

FOR $\ell = 1$ IS TRIVIAL

FOR $\ell > 1$: LET US DEFINE $d := HJ(k, \ell - 1)$

WE DEFINE A SEQUENCE OF NUMBERS

$$N_1 := k^{\ell^d} \quad N_2 := k^{\ell^{d+N_1}}, \quad N_3 := k^{\ell^{d+N_1+N_2}}, \quad \dots$$

$$N_i := k^{\ell^{d+\sum_{j=1}^{i-1} N_j}} \qquad \forall i \in \{1, \dots, d\}$$

$$N := N_1 + \dots + N_d .$$

ANY WORD $\in A^N$ CAN BE SPLIT

WE WILL CONSTRUCT A SEQUENCE:

$\sigma^{(1)}, \dots, \sigma^{(d)}$, WHERE

$\sigma^{(i)}$ IS A ROOT OF LENGTH N_i

$$X = X_1 \dots X_d \in A^d \rightsquigarrow \underbrace{\sigma^{(1)}(x_1) \dots \sigma^{(d)}(x_d)}_{\vec{\sigma}(X)} \in \hat{A}$$

$A = \{1, 2, 3\}$, $d = 2$

$\sigma^{(1)} := **1$

$\sigma^{(2)} := 2*$

$23 \dashrightarrow \vec{\sigma}(23) = 22123$

08.01.2008 Tu
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DM1012 KOMBINATORIKA A GRAFY II

IF $\{\rho(1), \dots, \rho(\ell)\} \subseteq A^d$ IS A COMB. LINE

$\Rightarrow \{\vec{\sigma}(\rho(1)), \dots, \vec{\sigma}(\rho(\ell))\} \subseteq A^N$ IS A COMB. LINE

(THIS SHOULD BE CLEAR)

$X = X_1 \dots X_d \in A^d$
$Y = Y_1 \dots Y_d \in A^d$

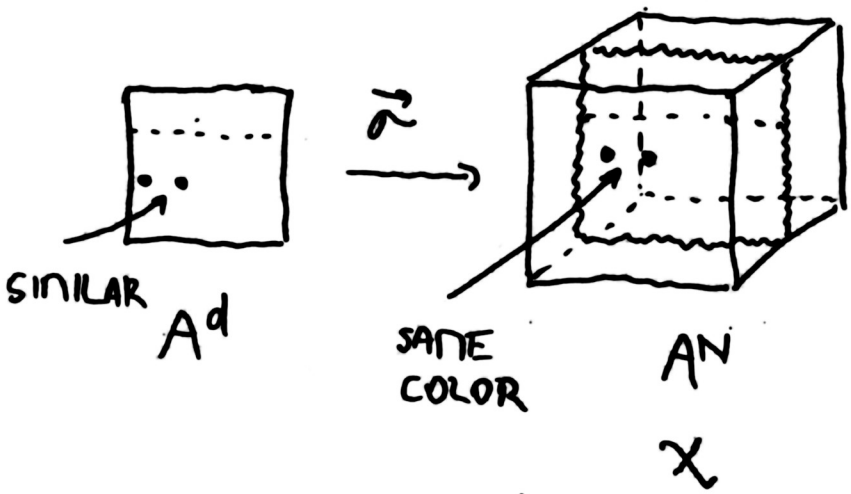
X, Y ARE "SIMILAR" IF FOR EVERY $i = 1, \dots, d$
EITHER $X_i = Y_i$
OR $X_i = 1$ AND $Y_i = 2$
OR $X_i = 2$ AND $Y_i = 1$

(1 AND 2 CANNOT BE DISTINGUISHED)

KEY CLAIM: LET US ASSUME THAT
$\chi : A^N \to \{1, \dots, k\}$

THERE ARE $\sigma(1), \dots, \sigma(d)$, WITH THE PROPERTY THAT FOR EVERY $X, Y \in A^d$ SIMILAR

$\chi(\vec{\sigma}(X)) = \chi(\vec{\sigma}(Y))$



LET $B := A \setminus \{1\} = \{2, \dots, \ell\}$

$B^d \subseteq A^d$

NOW WE DEFINE $\chi' : B^d \to \{1, \dots, k\}$

$x_1 \dots, x_d \in B^d$

$\chi'(x_1 \dots, x_d) := \chi(\vec{\sigma}(x_1 \dots x_d))$

BY INDUCTION : $\exists$ MONOCHROMATIC LINE :

$\{\rho(2), \dots, \rho(\ell)\} \subseteq B^d$

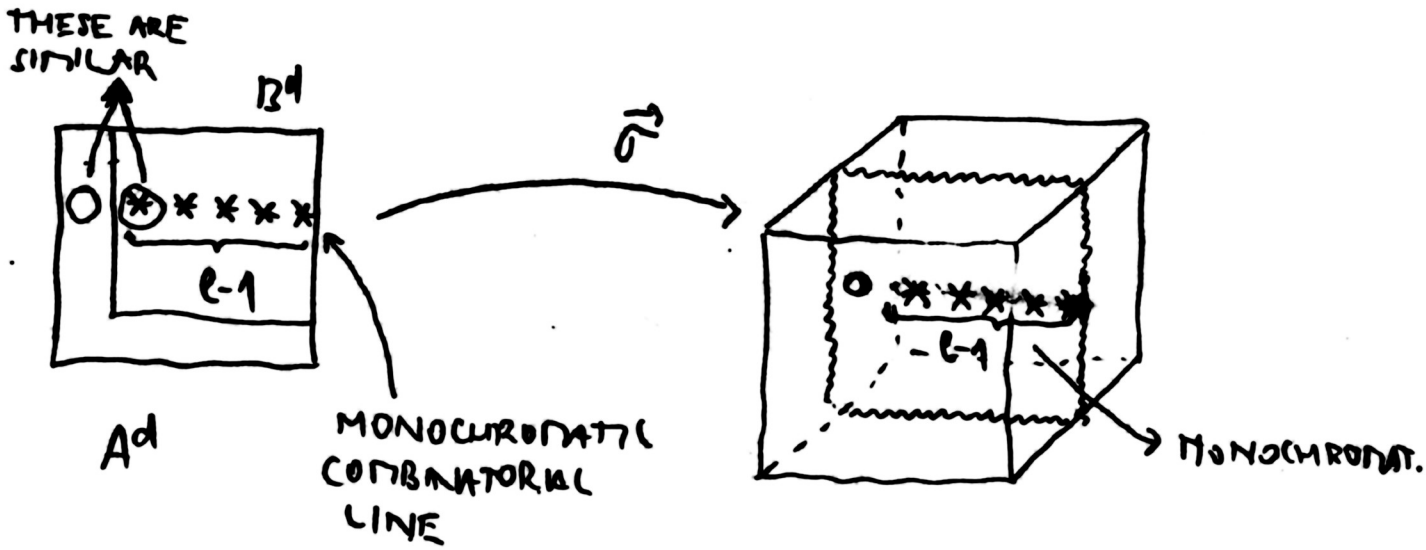
$\{\vec{\sigma}(\rho(2)), \dots, \vec{\sigma}(\rho(\ell))\} \subseteq A^N$ IS MONOCHROMATIC

NOW $\rho(1)$ AND $\rho(2)$ ARE SIMILAR

$\Rightarrow \vec{\sigma}(\rho(1))$ AND $\vec{\sigma}(\rho(2))$ HAVE THE SAME COLOR.

$\Rightarrow \{\vec{\sigma}(\rho(1)), \dots, \vec{\sigma}(\rho(\ell))\} \subseteq A^N$

IS MONOCHROMATIC LINE.



08.01.2008 Th
PETER ČERNO

DM1012 KOMB. A GRAFY II

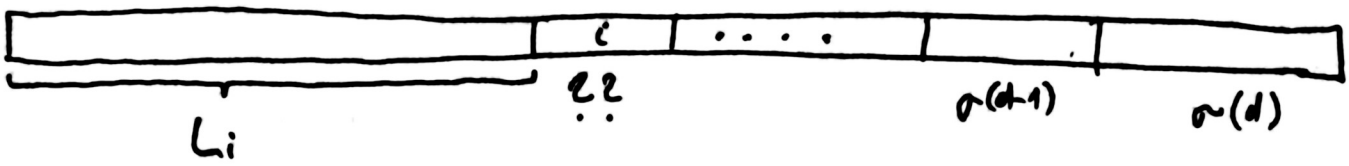
WANTED: $\sigma^{(1)}, \dots, \sigma^{(d)}$, $\sigma^{(i)}$ .. ROOT OF LEN. N_i

AND IF $x_1 \dots x_d$ AND $y_1 \dots y_d$ ARE SIMILAR
THEN $\vec{\sigma}(x_1 \dots x_d)$ AND $\vec{\sigma}(y_1 \dots y_d)$
HAVE THE SAME COLOR.

BY REVERSE INDUCTION:
$i \in \{1, \dots, d\}$, ASSUME
THAT $\sigma^{(i+1)}, \dots, \sigma^{(d)}$ ARE KNOWN
AND WE WANT TO CONSTRUCT $\sigma^{(i)}$

LET $L_i \overset{\text{def}}{:=} N_1 + \dots + N_{i-1}$.

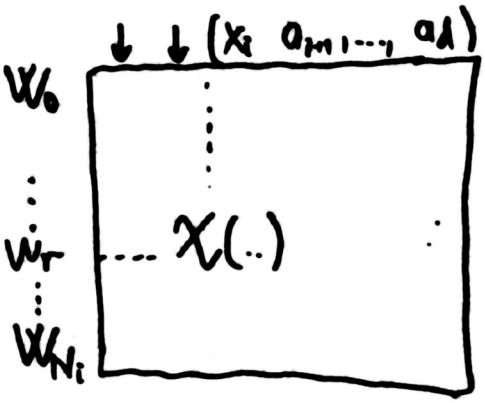
RECALL $N_i \overset{\text{def}}{:=} k^{\ell^{d+L_i}}$.



$W_0 := 111 \dots 11$
$W_1 := 111 \dots 12$
$W_2 := 111 \dots 122$
$\vdots$
$W_{N_i} = 222 \dots 22$

$\chi(\underbrace{X}_{L_i} W_r \, \sigma^{(i+1)}(a_{i+1}) \dots \sigma^{d}(a_d))$

IN THEESE WORDS WE ARE INTERESTED



$N_i + 1$ ROWS.

$\ell^{L_i + d - 1}$ COLUMNS

MATRIX OF COLORS

THE MAIN OBSERVATION : THIS MATRIX SHOULD CONTAIN TWO IDENTICAL ROWS.

BY PIGEONHOLE PRINCIPLE

IT CAN HAVE AT MOST $k^{\ell L_i + d - i}$ DISTINCT ROWS

BUT IT HAS $N_i + 1$ ROWS $> k^{\ell L_i + d - i}$.

LET SAY THAT W_r, W_s $r < s$ ARE IDENTICAL ROWS.

NOW WE CAN FINALLY DEFINE

$$\sigma^{(i)} := \underbrace{11\ldots1}_{N_i - s}\ \underbrace{**\ldots*}_{s-r}\ \underbrace{22\ldots2}_{r}$$

$$\sigma^{(i)}(1) = \underbrace{11\ldots1}_{N_i - r}\ \underbrace{22\ldots2}_{r} = W_r$$

$$\sigma^{(i)}(2) = \underbrace{11\ldots1}_{N_i - s}\ \underbrace{22\ldots2}_{s} = W_s .$$

IF

$x = x_1 \ldots x_d$
$y = y_1 \ldots y_d$

ARE SIMILAR, WLOG x AND y DIFFER IN A SINGLE SYMBOL i

$x_i = 1$, $y_i = 2$.

$$\chi(\vec{\sigma}(x)) = \chi(\sigma^{(1)}(x_1) \ldots \sigma^{(i)}(1) \ldots \sigma^{(d)}(x_d))$$
$$\chi(\vec{\sigma}(y)) = \chi(\sigma^{(1)}(x_1) \ldots \sigma^{(i)}(2) \ldots \sigma^{(d)}(x_d))$$

$$\chi(\vec{\sigma}(x)) = \chi(\ldots W_r\, \sigma^{(i+1)}(x_{i+1}) \ldots \sigma^{(d)}(x_d)) =$$
$$= \chi(\vec{\sigma}(y)) = \chi(\ldots W_s\, \sigma^{(i+1)}(x_{i+1}) \ldots \sigma^{(d)}(x_d)) . \blacksquare$$

08.01.2008 Tu
PETER ČERNO

EX: DM1012 KOMB. A GR. II

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CORROLLARY [VAN DER WAERDEN]

$\forall k, \ell \, \exists M$: EVERY k-COLORING OF $\{0, 1, 2, \dots, M\}$ CONTAINS A MONOCHROMATIC ARITHMETIC PROGRESSION OF LENGTH ℓ.

PROOF: $x, \; x+d, \; x+2d, \dots, \; x+(\ell-1)d \qquad x, d \in \mathbb{N}$

$M := HJ(k, \ell) \cdot \ell$

$N := HJ(k, \ell)$

NOW LET $\chi : \{0, \dots, M\} \to \{1, \dots, k\}$

$A = \{1, \dots, \ell\}$

AND DEFINE $\bar{\chi} : A^N \to \{1, \dots, k\}$

$\bar{\chi}(a_1 \dots a_N) := \chi(a_1 + \dots + a_N)$

HJT: $\exists \; \{\rho(1), \dots, \rho(\ell)\}$ IS MONOCHROMATIC COMBINATORIAL LINE:

$\rho := a_1 \dots a_j * \dots *$ (THIS WE CAN ASSUME)

WE HAVE

$$a_1 \dots a_j \, 11 \dots 1 \rightsquigarrow a_1 + a_2 + \dots + a_j + \overbrace{1 + 1 + \dots + 1}^{d}$$

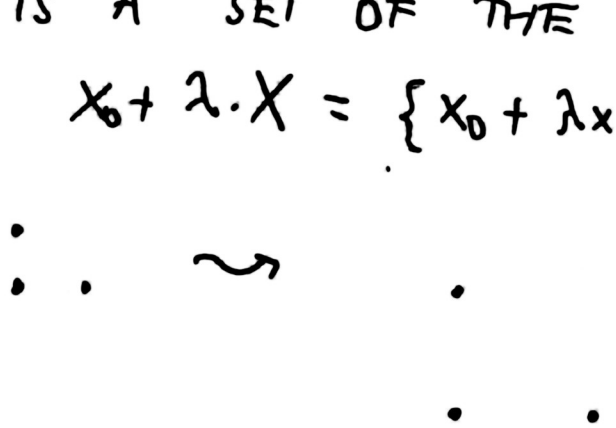
$$\vdots$$

$$a_1 \dots a_j \, \ell\ell \dots \ell \rightsquigarrow a_1 + a_2 + \dots a_j + \ell + \ell + \dots + \ell$$

WITH THE SAME COLOR

$\mathbb{Z}^d$, $X \subseteq \mathbb{Z}^d$, HOMOTHETIC COPY OF X IS A SET OF THE FORM

$$x_0 + \lambda \cdot X = \{x_0 + \lambda x : x \in X\} \qquad x_0 \in \mathbb{Z}^d, \lambda \in \mathbb{N}$$



ARITHMETIC PROGRESSION IS A HOMOTHETIC COPY OF $\{1, \dots, \ell\}$.

COROLLARY (GALLAI, WITT): $\forall X \subseteq \mathbb{Z}^d$, X FINITE $\forall k$ EVERY k-COLORING OF $\mathbb{Z}^d$ HAS A MONOCHROMATIC HOMOTHETIC COPY OF X.

PROOF: $N := HJ(k, \ell)$ WHERE $\ell := |X|$

$X = \{v_1, \dots, v_\ell\}$

X.. ALPHABET

THE ENEMY CHOOSES $\chi : \mathbb{Z}^d \to \{1, \dots, k\}$

$\bar{\chi} : X^N \to \{1, \dots, k\}$

$$\bar{\chi}(x_1, \dots, x_N) := \chi(x_1 + \dots + x_N)$$

FOR $\rho = a_1 \dots a_j * \dots *$ WE HAVE $\boxed{a_1 + \dots + a_j + (N-j) X}$

$$\left.\begin{array}{l} a_1 \dots a_j v_1 \dots v_1 \rightsquigarrow a_1 + \dots + a_j + v_1 + \dots + v_1 \\ \dots \\ a_1 \dots a_j v_\ell \dots v_\ell \rightsquigarrow a_1 + \dots + a_j + (N-j) v_\ell \end{array}\right\}$$

$\square$

08.01.2008 Th
PETER ČERNO

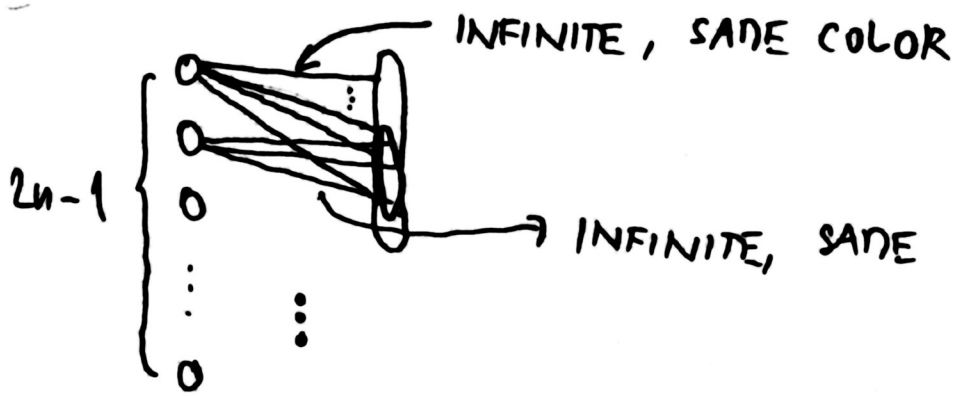
EX: DM1012 KOMB. A GRAFY II

1) IS IT TRUE $\forall$ 2-COLORING OF EDGES OF $K_{\mathbb{N},\mathbb{N}}$ a) CONTAINS SOME MONOCHROMATIC $K_{n,n}$ $\forall n$

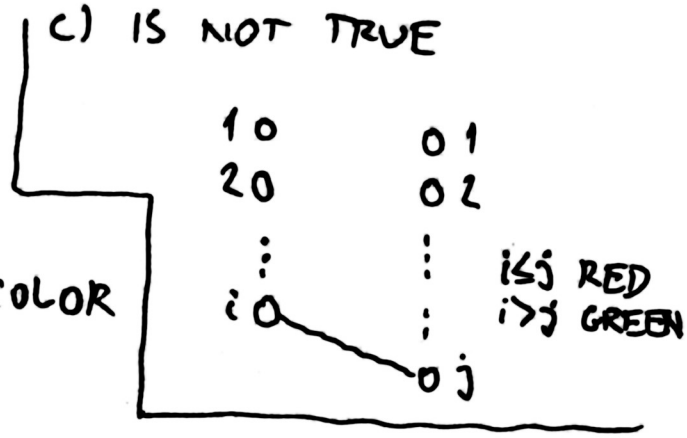
b) $K_{n,\infty}$

c) $K_{\infty,\infty}$

a) AND b) ARE CLEAR



c) IS NOT TRUE



2) COUNTABLE DILWORTH: $\forall$ COUNTABLE POSET $(X,\preceq)$ $\forall k \in \mathbb{N}$: X CAN BE COVERED BY k CHAINS $\Leftrightarrow$ X HAS NO ANTICHAIN LARGER THAN k.

3) $HJ(k, 2) \leq k$

k: #COLORS, 2: #SYMBOLS

EVERY k-COLORING OF $\{1,2\}^k$ CONTAINS A MONOCHROMATIC COMBINATORIAL LINE

FOR $k=1$ TRIVIAL, WE HAVE 1 COLOR

FOR $k > 1$, WE HAVE

$$x : \{1,2\}^k \to \{1, \dots, k\}$$

~~IF WE FIX 1, WE HAVE WORDS IN $\{1,2\}^{k-1}$~~

$$\begin{array}{c|c|c|c|c|c|c|c}
 & & & & \{1,2\}^{k-1} & & & \\
\hline
1 & & & & \dots & & & \\
\hline
2 & & & & \dots & & & \\
\hline
 & & & & \dots & & &
\end{array}$$

2 ROWS

2^{k-1} COLUMNS

$\downarrow$

k^2 POSSIBILITIES

IF WE TAKE ~~AN ARBITRARY~~

FOR INSTANCE

$$\begin{array}{ccc}
11 & \dots & 1 \\
21 & \dots & 1 \\
22 & \dots & 1 \\
\vdots & & \\
22 & \dots & 2
\end{array}$$

02.01.2008 T4
PETER ČERNO

EX: DM1012 KOMB. A GRAFY II

A "POSITIVE DENSITY HALES ~ JEWETT" 1990's

$\forall \ell \in \mathbb{N} \;..\; |A| = \ell \quad \forall \varepsilon > 0 \;\exists N = N(\varepsilon, \ell)$

$\forall X \subseteq A^N \;..\; |X| \geq \varepsilon |A^N|$

X CONTAINS A ~~MONOCHROMATIC~~ COMB. LINE.

PROVE $\Rightarrow$ HJT

LET US TAKE $k \in \mathbb{N}$. LET $\varepsilon := \frac{1}{k}$

06.12.2007 Sa
PETER ČERNO

DM1012 KOMBINATORIKA A GRAFY II HW

ÚLOHA 1: Je pre každú $f: \mathbb{N} \to \{0,1\}$ existuje aritmetická postupnosť $\{a_i\}_{i=1}^{\infty}$: $f(a_i) = \text{konst.}$?
$a_i = a_0 + k \cdot i$, $a_0, k \in \mathbb{N}$.

STAČÍ UVAŽOVAŤ

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16 ...
$f(n)$	0	1	1	0	0	0	0	1	1	1	1	1	1	1	1	0 ...

$$f(n) = \frac{1-(-1)^{\lfloor \log_2 n \rfloor}}{2}$$

DĹŽKA JE $2^{\lfloor \log_2 n \rfloor}$

UVAŽUJME NEJAKÉ $a_0, k \in \mathbb{N}$. STAČÍ NÁJSŤ TAKÉ $i, j \in \mathbb{N}$, ŽE $\lfloor \log_2(a_0 + ki) \rfloor$ A $\lfloor \log_2(a_0 + kj) \rfloor$ MAJÚ OPAČNÚ PARITU.

NECH $c, d \in \mathbb{N}$ $2^c \leq a_0 < 2^{c+1}$ $2^d \leq k < 2^{d+1}$

~~POLOŽME $i := 2^c$, $j := 2^{c+1}$~~
~~$2^{c+d} < a_0 + 2^d \cdot 2^c < a_0 + k 2^c < a_0 + 2^{c+d+1}$~~

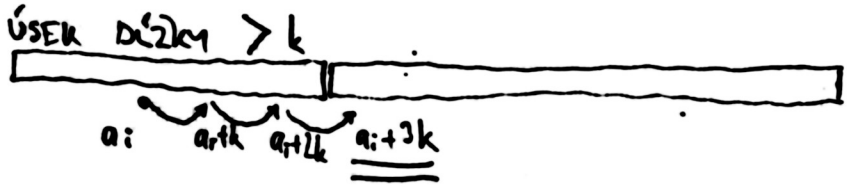
~~POLOŽME $i := 2^c$ TAK, ABY $2^c \leq a_0 + 2^d \cdot 2^c$~~

ZVOLÍME i TAK, ABY $k < \lfloor \log_2 a_i \rfloor = \lfloor \log_2 a_0 + ki \rfloor$

NAPR. $\underline{\underline{i := 2^{k+2}}}$ $2^{k+1} < a_0 + k 2^{k+2}$

$$\Rightarrow \quad k+1 < \log_2(a_0 + k 2^{k+2})$$
$$\Rightarrow \quad k < k+1 \leq \lfloor \log_2 a_0 + k i \rfloor$$

ÚSEK DĹŽKY $> k$



ÚLOHA 2: PRE $G=(V,E)$ MOŽNO HRANY ORIENTOVAŤ TAK, ŽE $\max_{v\in V_G} \deg^+_G v \le k$

$\Leftrightarrow \forall H \subseteq G \quad |E_H| \le k|V_H|$

NECH $\vec{H} \subseteq \vec{G}$: $|E_H| > k|V_H|$, TJ. $\sum_{v\in V_H} \deg^+_H v > k|V_H|$

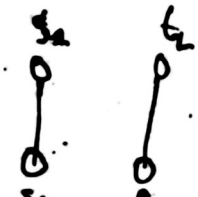
$\Rightarrow$ NUTNE EXISTUJE VRCHOL $v \in V_H$: $\deg^+_H v > k$.

$\Rightarrow$) NAOPAK PRE KAŽDE $\vec{H} \subseteq \vec{G}$ PLATÍ $\max_{v\in V_H} \deg^+_H v \le k$

$\Rightarrow \quad |E_H| = \sum_{v\in V_H} \deg^+_H v \le k|V_H|$.

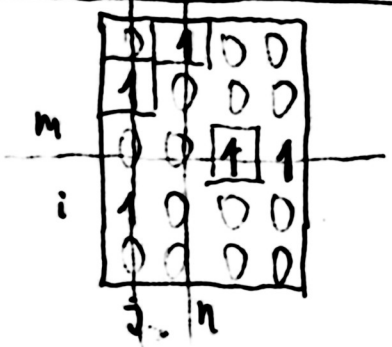
ÚLOHA 3: NECH $M \in \{0,1\}^{m\times n}$

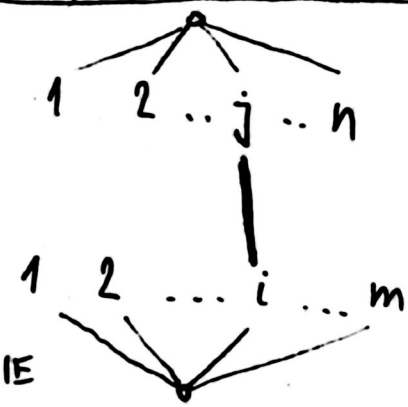
NEZÁVISLÁ MNOŽINA $S \subseteq \{1,\dots,m\}\times\{1,\dots,n\}$ MÁ VLASTNOSŤ $\forall s,t \in S \quad s_1 \ne t_1 \wedge s_2 \ne t_2$



ŽIADNE 2 HRANY NEMAJÚ SPOL. VR.

$\forall s \in S \; M_s \ne 0$

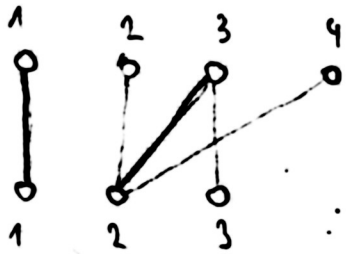


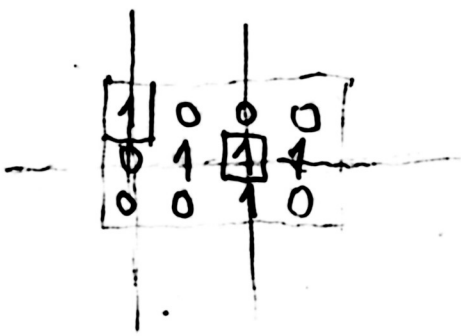


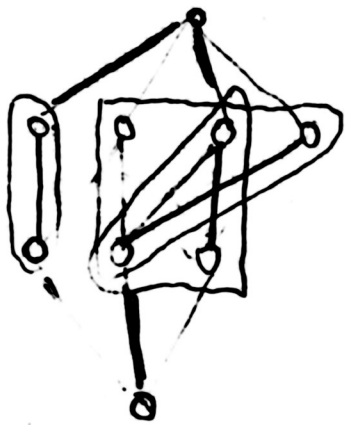
POKR. ČIAR MUSÍ BYŤ ASPOŇ TOLKO, KOĽKO MÁ MAX. NEZ. MNOŽINA PRVKOV, KAŽDÁ ČIARA POKRÝVA MAX. JEDEN PRVOK MNOŽINY S.

NEZÁV. MN. $\equiv$ PÁROVANIE

$\Rightarrow$ MAX. NEZÁVISLÁ MN. JE MAX. PÁROVANIE







06.10.2007 Sa
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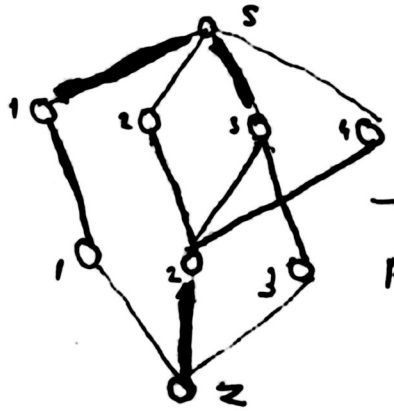
DM1012 KOMB. A GRAFY II HW

	1	2	3	4
1	1	0	0	0
2	0	1	1	1
3	0	0	1	0



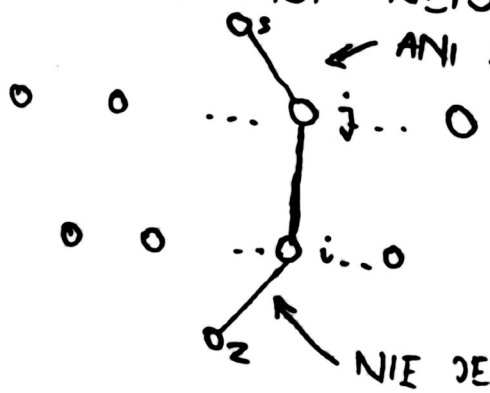
STĹPCE

→ JEDNOTKY SÚ OZNAČENÉ HRANAMI

RIADKY

PODĽA MAX FLOW - MIN CUT THEOREM EXISTUJE REZ VEĽKOSTI |S|, BÚNO NECH HRANY TOHTO REZU VYCHÁDZAJÚ VŽDY BUĎ ZO Z ALEBO S.

KEĎ EXISTOVALA NEPOKRYTÁ 1



ANI STĹPEC

NAŠLI BY SME CESTU ZO Z DO S

NIE JE SÚČASŤOU REZU, TJ. NIE JE ŠKRTNUTÝ i-TÝ RIADOK

ÚLOHA 4:

1	2	3
1 2 3	4 5 6	7 8 9
4 5 8	1 9 7	2 6 3
A_{11}	A_{21}	A_{31}

$M = \{0, 1, 2$

A_1, A_2, A_3 MAJÚ 3 PRVKY A SÚ NAVZÁJOM DISJUNKTNÉ

KEĎ $a \in A_i \cap A_j$ $i \neq j$, POTOM a SA NEVYSKYTUJE V CHLIEVIKU i ANI j => NUTNE 2x V CHLIEVIKU k

$M = \{0, 1, 2, 3, \quad , \quad , 6,$

VETA O LAT. OBDÉLNÍKU

1 2 3	4 5 6	7 8 9
4 5 6	7 8 9	1 2 3
7 8 9	1 2 3	4 5 6
2 3 5	6 9 7	8 1 4
0		

5 1 1	2 4 1	2 3 2
6 6 5	3 3 3	3 4 4
8 7 7	1 4 4	5 6 5
9 8 8	9 9 8	6 7 7

6	1	1	2	1	1	2	3	2
8	4	4	3	3	2	3	4	5
9	6	7	5	4	4	5	6	7
	7	8	6	8	5	6	7	8
	9				8	5	9	

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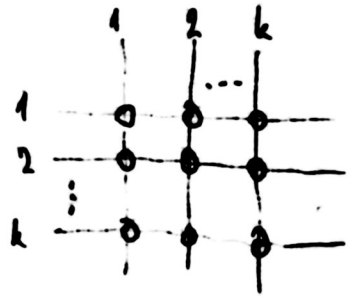
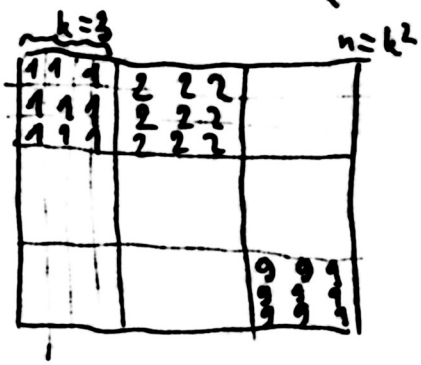
DM1012 KOMBINATORIKA A GRAFY II

PROBLEM 4: $M \in \{1,\dots,n\}^{n\times n}$ $\forall i \in \{1,\dots,n\}$ APPEAR IN M n-TIMES.

a) SUPPOSE THAT EACH ROW AND COLUMN HAS AT MOST $k = \lceil\sqrt{n}\rceil - 1$ DIFFERENT NUMBERS. $k^2 < n$.

LET'S TAKE ANY NUMBER $i \in \{1,\dots,n\}$
THIS NUMBER FALLS INTO AT LEAST $\lceil\sqrt{n}\rceil = k+1$ ROWS OR COLUMNS, BECAUSE IF IT IS IN AT MOST k ROWS AND IN AT MOST k COLUMNS, THEN i OCCURS IN M AT MOST $k^2 < n$ TIMES ↯.

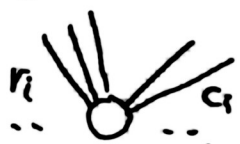




LET r_i, c_i BE THE NUMBER OF ROWS, COLUMNS, THAT i FALLS INTO. THEN $r_i \cdot c_i \geq n$. $\forall i \in \{1,\dots,n\}$.

~~LET R_k^i BE THE NUMBER OF i IN k-TH ROW~~
~~C_k^i BE THE NUMBER OF i IN k-TH COLUMN.~~
~~$\sum_{i=1}^{n} R_k^i$ IS THE NUMBER OF DIFFERENT NUMBERS IN k-TH ROW.~~
~~$\sum_{k=1}^{n} R_k^i = \sum_{k=1}^{n} C_k^i \geq n$. $\forall i \in \{1,\dots,n\}$~~

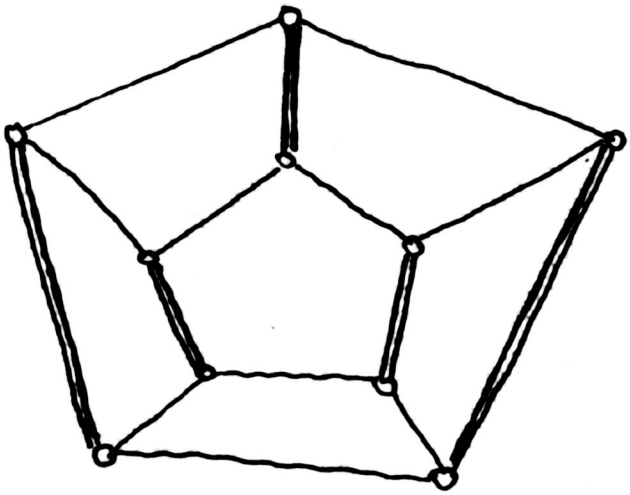
SUPPOSE $R_i, C_i \leq k$ $\forall i$
$\Rightarrow$ THE NUMBER OF EDGES $m \leq k \cdot 2n < \sqrt{n}\,2n$

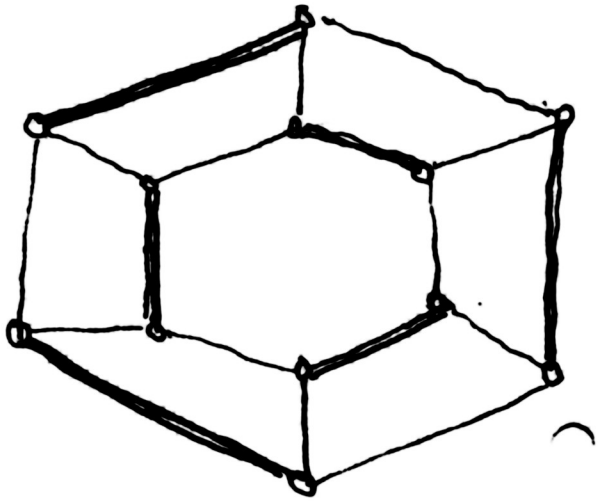


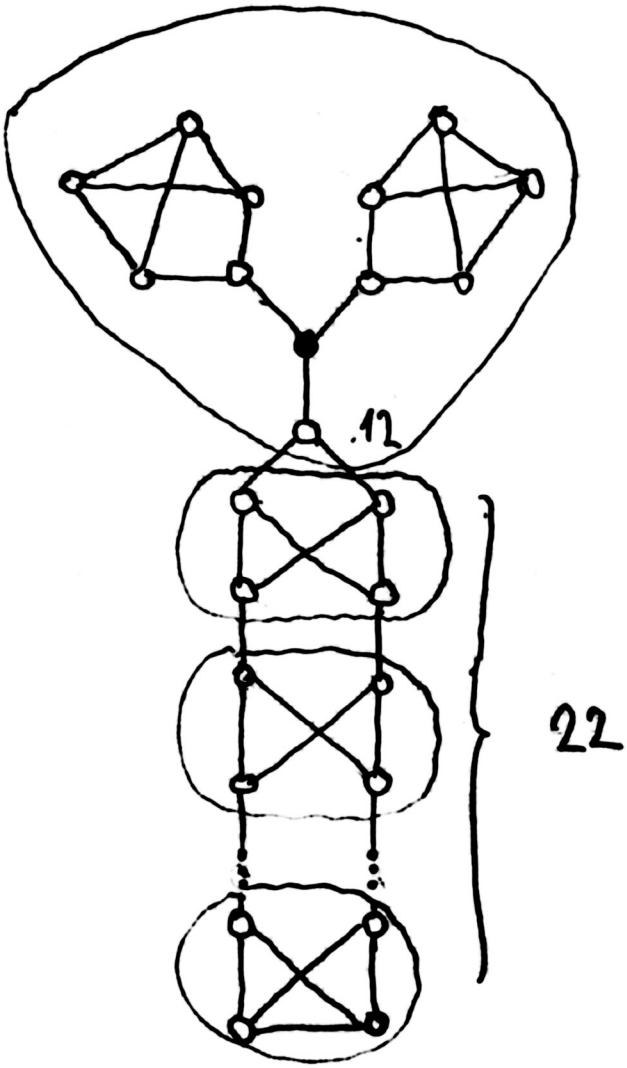
$m = \sum_{i=1}^{n} r_i + c_i$ $\quad r_i c_i \geq n$

AG: $\frac{m}{2n} \geq \sqrt[2n]{\prod r_i c_i} \geq \sqrt[2n]{n^n} = \sqrt{n} \Rightarrow m \geq \sqrt{n}\,2n$ □

PROBLEM 5 :







$22.4 + 12 = 100$

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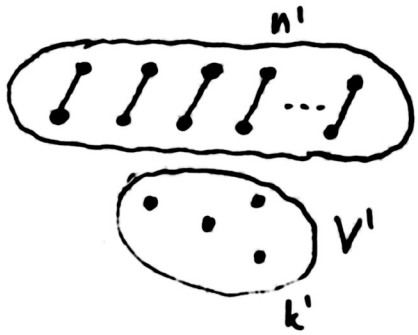
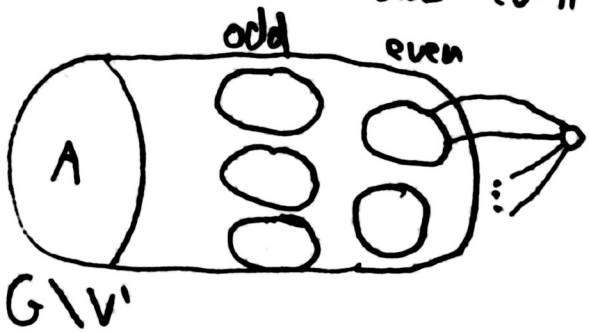
DM1012 KOMB. a GRAFY II

PROBLEM 6 :

a) $\Rightarrow$) SUPPOSE $G=(V,E)$ HAS A MATCHING THAT COVERS $n' = |V| - k' \geq |V| - k$ VERTICES ($2 \mid n'$, $k' \leq k$)



APPARENTLY $G \setminus V'$ HAS A PERFECT MATCHING SO FOR EVERY $A \subseteq V \setminus V'$ $(G \setminus V') \setminus A$ HAS AT MOST $|A|$ ODD COMPONENTS.



ADDING ONE VERTEX WITH SOME EDGES CAN ADD AT MOST ONE ODD COMPONENT AND THE GRAPH $G \setminus A$ CAN BE CONSTRUCTED FROM $(G \setminus V') \setminus A$ BY ADDING SOME VERTICES AND EDGES (AT MOST k' VERTICES),

SO $c_{odd}(G \setminus A) \leq c_{odd}((G \setminus V') \setminus A) + k' \leq |A| + k' \leq |A| + k$

$\Leftarrow$) LET'S ASSUME THAT FOR EACH SET $A \subseteq V$
$C_{odd}(G \setminus A) \leq |A| + k$. WE WANT TO PROVE THAT
G HAS A MATCHING THAT COVERS AT LEAST $|V| - k$ VERTICES.

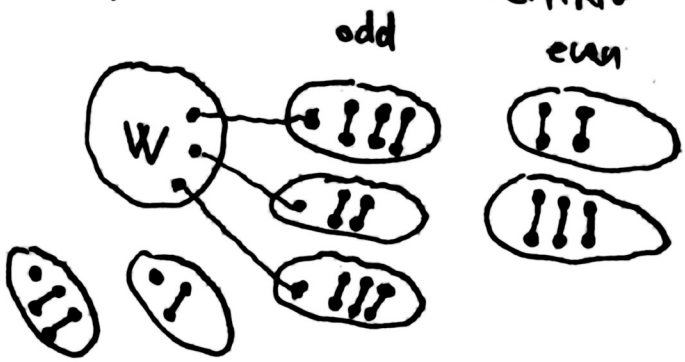
INDUCTION ON THE NUMBER OF EDGES (FROM $\binom{n}{2}$ TO 0)

I. K_n, $A = \emptyset \Rightarrow C_{odd}(K_n) \leq k$ $\boxed{\text{WE CAN ASSUME } k \geq 1}$

FOR $k = 0 \Rightarrow$ PERFECT MATCHING
FOR $k > 0$ IF $2 \nmid n \Rightarrow$ MATCHING THAT COVERS $|V| - 1$ VER.

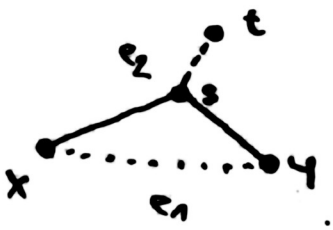
II. $G \neq K_n$, $W := \{u, \deg_G u = n-1\}$

a) $G \setminus W$ IS DISJOINT UNION OF COMPLETE GRAPHS
$C_{odd}(G \setminus W) \leq |W| + k$
EACH ODD COMPONENT IN $G \setminus W$ HAS AN UNPAIRED VERTEX THAT WE CAN PAIR WITH ONE VERTEX IN W
EXCEPT AT MOST k ODD COMPONENTS
$\Rightarrow$ WE HAVE MATCHING THAT COVERS AT LEAST $|V| - k$ VER.



$k = 2$

b) $G \setminus W$ IS NOT DISJOINT UNION OF COMPLETE GRAPHS
$\Rightarrow \exists$ INDUCED $K_{1,2}$



..... M_1
- - - M_2

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BY INDUCTION IN $G+e_1$, $G+e_2$ THERE EXIST A MATCHING M_1, M_2 THAT COVERS AT LEAST $|V|-k$ VERTICES

(WE CAN ASSUME $e_1 \in M_1$, $e_2 \in M_2$)

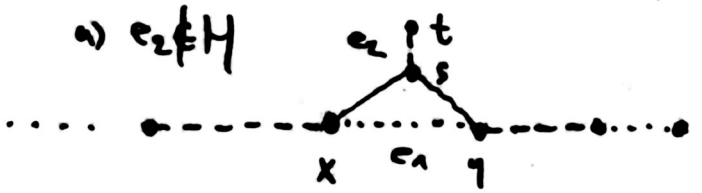
$M := M_1$ XOR M_2

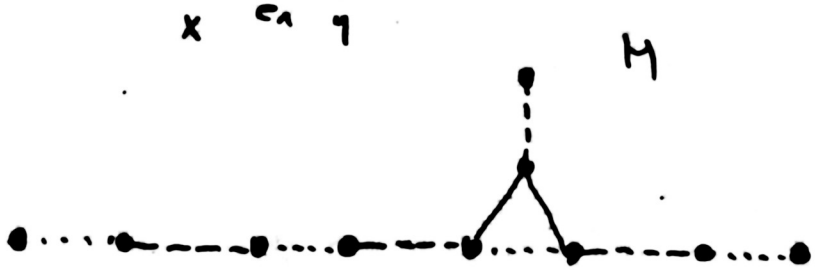
$\deg_M u = \begin{cases} 0 \\ 1 \\ 2 \end{cases}$, GRAPH (V, M) CONSISTS OF ISOLATED VERTICES, PATHS AND EVEN CYCLES

I. LET H BE A CYCLE, $e_1 \in H$ $\Rightarrow$ THE SAME PROOF AS IN TUTTE

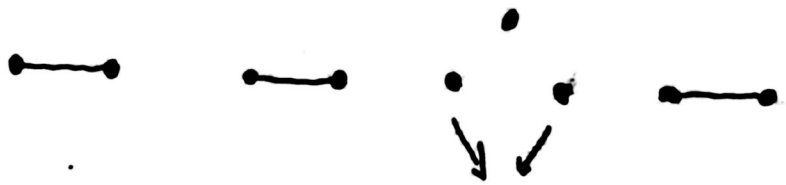
II. LET H BE A PATH, $e_1 \in H$

a) $e_2 \notin H$





$\Downarrow$ WE CHOOSE $M_1 \setminus e_1$

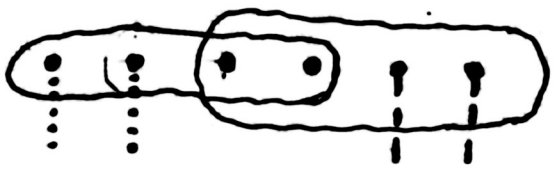


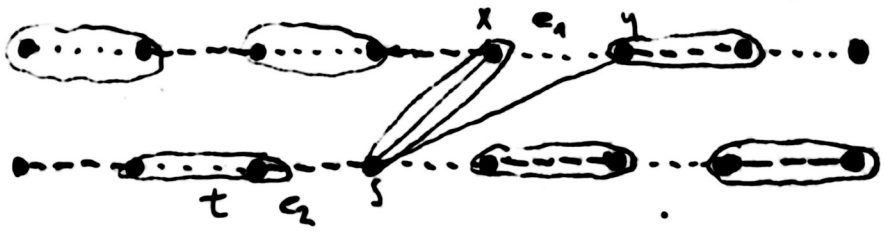
2 NEW UNPAIRED VERTICES !

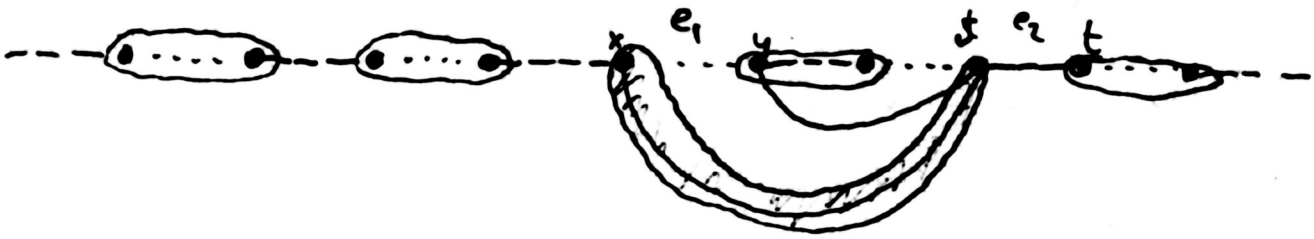
M_1 AND M_2 COVER AT LEAST $|V|-k$. WE CAN ASSUME THAT M_1 AND M_2 COVER EXACTLY $|V|-k' \geq |V|-k$ VERTICES.

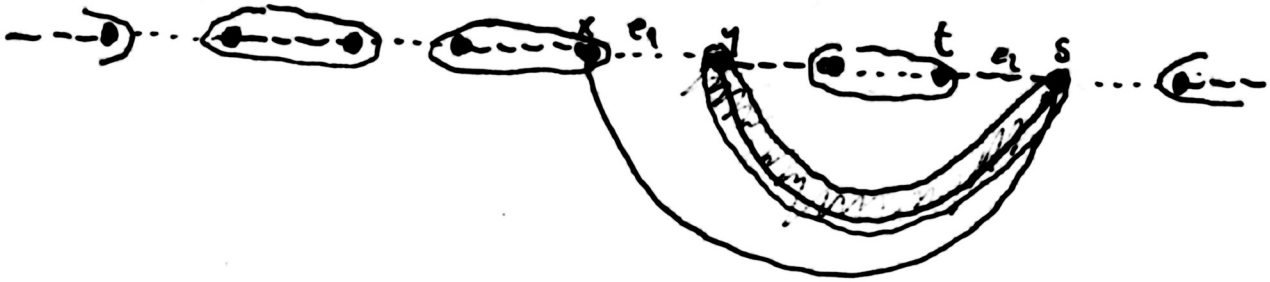
. IF THEY DO NOT COVER THE SAME VERTICES $\Rightarrow \deg u \begin{cases} 0 \\ 2 \end{cases}$

IT IS THE SAME AS IN THE TUTTE THEOREM.




x
e_1
y
t
e_2
s


x
e_1
y
s
e_2
t


x
e_1
y
t
e_2
s

$\Sigma = 18$

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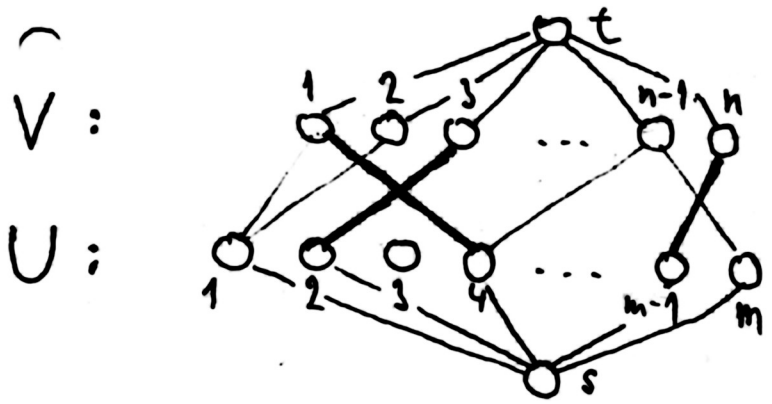
DM1012 HOMEWORK

PROBLEM 3: LET $M \in \{0,1\}^{m \times n}$.

AN INDEPENDENT SET S OF M IS $S \subseteq [m] \times [n]$ SUCH THAT $(\forall s,t \in S : s \neq t \Rightarrow s_1 \neq t_1 \wedge s_2 \neq t_2) \wedge (\forall s \in S\ M_{s_1,s_2} = 1)$

IF WE DRAW A GRAPH G THAT REPRESENTS A FLOW NETWORK WITH SOURCE s AND SINK t



EXAMPLE FOR $S = \{(2,3), (1,4), \ldots \ldots (m-1, n)\}$

G IS DEFINED:
$(u,v) \in G \Leftrightarrow M_{u,v} = 1 \quad u,v \neq s,t$

AND IF WE (DRAW) CHOOSE AN EDGE (u,v) FOR ALL $(u,v) \in S$, THEN WE GET A MATCHING. IT IS BECAUSE THE DEFINITION: $s \neq t \Rightarrow s_1 \neq t_1 \wedge s_2 \neq t_2$ SAYS THAT NO TWO DIFFERENT EDGES HAVE A COMMON VERTEX.

ON THE OTHER HAND FOR EACH MATCHING IN G WE CAN EASILY CONSTRUCT THE CORRESPONDING IND. SET S.

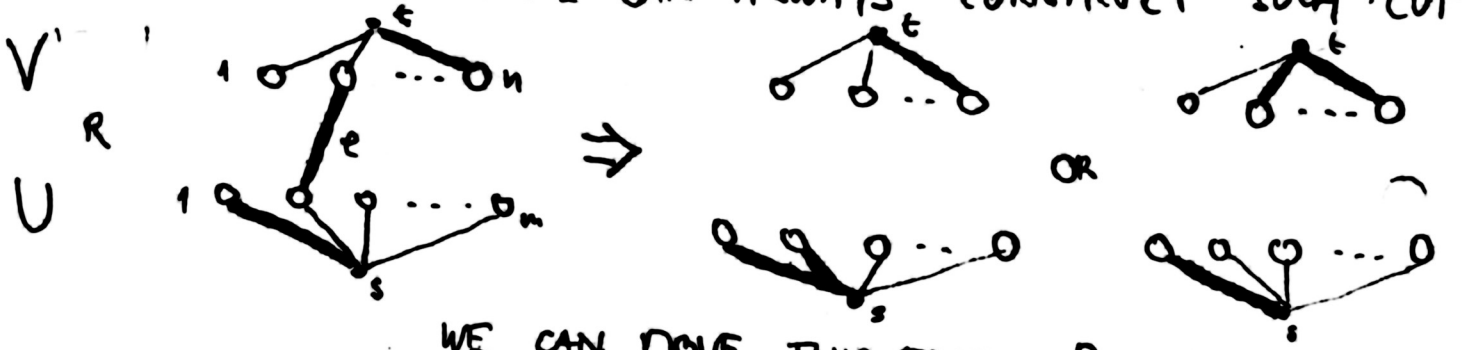
OBSERVATION: THE CARDINALITY OF A COVERING SET OF LINES IS AT LEAST THE CARDINALITY OF AN INDEPENDENCE SET. $(|L| \geq |S|)$

PROOF: EACH LINE IN L COVERS AT MOST ONE $s \in S$.

LET f_0 BE THE MAXIMUM FLOW IN G.
THIS FLOW DETERMINES THE MAXIMUM MATCHING.
LET S_0 BE A CORRESPONDING INDEPENDENCE SET FOR THIS MATCHING. APPARENTLY S_0 IS THE LARGEST INDEPENDENT SET OF M, IE

$$|S_0| = \max_{S \text{ IS AN INDEPENDENT SET OF } M} |S|$$

FROM THE MAX FLOW - MIN CUT THEOREM FOLLOWS THAT THERE EXISTS A CUT R_0 IN G SUCH THAT $|R_0| = |S_0|$, AND THIS CUT R_0 IS MINIMAL.
WE CAN ASSUME THAT ALL EDGES IN R_0 ARE ADJACENT TO THE SOURCE s OR SINK t.
IF IT IS NOT SO, WE CAN ALWAYS CONSTRUCT SUCH CUT



WE CAN MOVE THIS EDGE $e \in R$ UP OR DOWN AND IT WILL ALWAYS BE A CUT

NOW WE CAN CONSTRUCT A COVERING SET OF LINES L SUCH THAT $|L| = |R|$.

FOR EACH $\{s, u\} \in R$ WE ADD TO L A ROW u
AND FOR EACH $\{v, t\} \in R$ WE ADD TO L A COLUMN v.

APPARENTLY A ROW u COVERS ALL THE EDGES IN G (VERTICES IN $M = 1$) THAT ARE ADJACENT TO u (THE SAME FOR THE COLUMN v)
R IS A CUT, SO L COVERS ALL THE EDGES OF G.
$\Rightarrow$ L COVERS ALL $s \in [m] \times [n]$ SUCH THAT $M_{s_1, s_2} = 1$
AND L IS APPARENTLY MINIMAL (OTHERWISE WE GET A SMALLER CUT) ■

4/4

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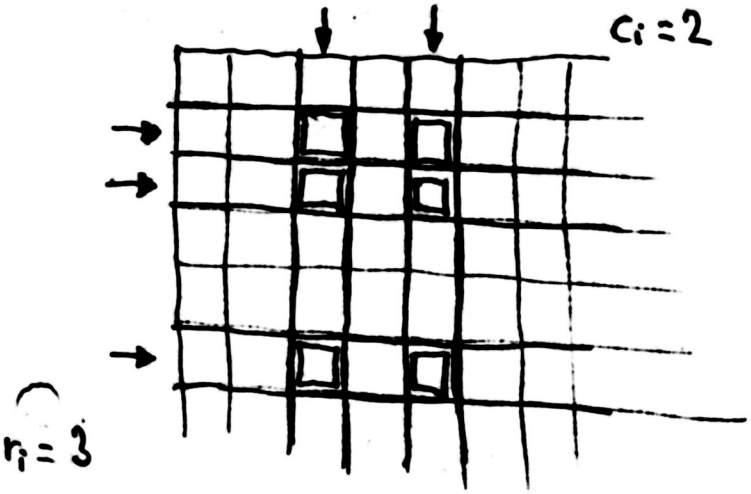
DM1012 HOMEWORK

PROBLEM 4:

LET $M \in [n]^{n \times n}$ AND $\forall i \in [n]$ i APPEARS IN M EXACTLY n-TIMES.

$\boxed{a)}$ SUPPOSE (FOR CONTRADICTION) THAT EACH ROW AND COLUMN HAS AT MOST $\lceil \sqrt{n} \rceil - 1 = k$ DIFFERENT NUMBERS (IE $k^2 < n$, BECAUSE $\lceil \sqrt{n} \rceil - 1 < \sqrt{n}$)

LET r_i BE THE NUMBER OF ROWS THAT CONTAIN i AND c_i BE THE NUMBER OF COLUMNS THAT CONTAIN i.



APPARENTLY THE NUMBER i CAN BE PLACED ONLY IN THE MARKED SQUARES

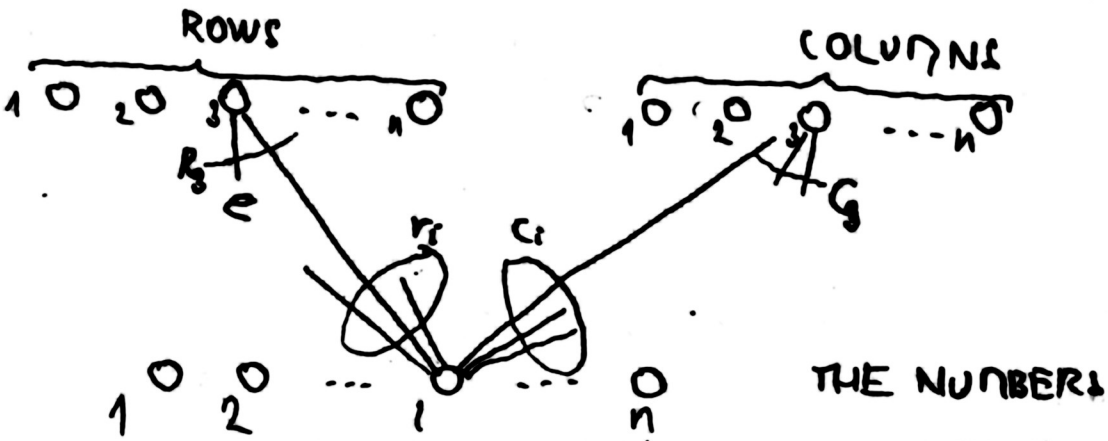
SO WE HAVE <u>AT MOST</u>

$r_i \cdot c_i$

POSITIONS FOR THE NUMBER i.

$$\Rightarrow \forall i \in [n] : \underline{\underline{r_i \cdot c_i \geq n}}.$$

NOW WE DROW A BIPARTITE GRAPH G:



AN EDGE $\{i, \text{ROWS}_j\}$ MEANS THAT THE NUMBER i OCCURS IN THE ROW j.

AN EDGE $\{i, \text{COLUMNS}_j\}$ MEANS THAT THE NUMBER i OCCURS IN THE COLUMN j.

LET R_j BE THE NUMBER OF EDGES ADJACENT TO ROW$_j$

AND C_j BE THE NUMBER OF EDGES ADJACENT TO COLUMN$_j$.

APPARENTLY THE TOTAL NUMBER m OF EDGES IN G IS : $m = \sum_{i=1}^{n}(r_i + c_i) = \sum_{i=1}^{n}(R_i + C_i)$.

SUPPOSE THAT $R_j, C_j \leq k = \lceil \sqrt{n} \rceil - 1 \quad \forall j \in [n]$.

$$\Rightarrow \quad m = \sum_{i=1}^{n}(R_i + C_i) \leq k \cdot 2n < \underline{\underline{2n\sqrt{n}}}. \qquad (01)$$

ON THE OTHER HAND $m = \sum_{i=1}^{n}(r_i + c_i) \quad r_i c_i \geq n$

AND FROM THE AG (ARITHMETIC - GEOMETRIC) INEQUALITY WE GET

$$\frac{m}{2n} = \frac{\sum_{i=1}^{n}(r_i + c_i)}{2n} \geq \sqrt[2n]{\prod_{i=1}^{n} r_i \cdot c_i} \geq \sqrt[2n]{n^n} = \sqrt{n}$$

$$\Rightarrow \quad m \geq 2n\sqrt{n} \qquad (02)$$

WHAT IS A CONTRADICTION WITH (01). $\lightning$ ✓ 4/4 ■

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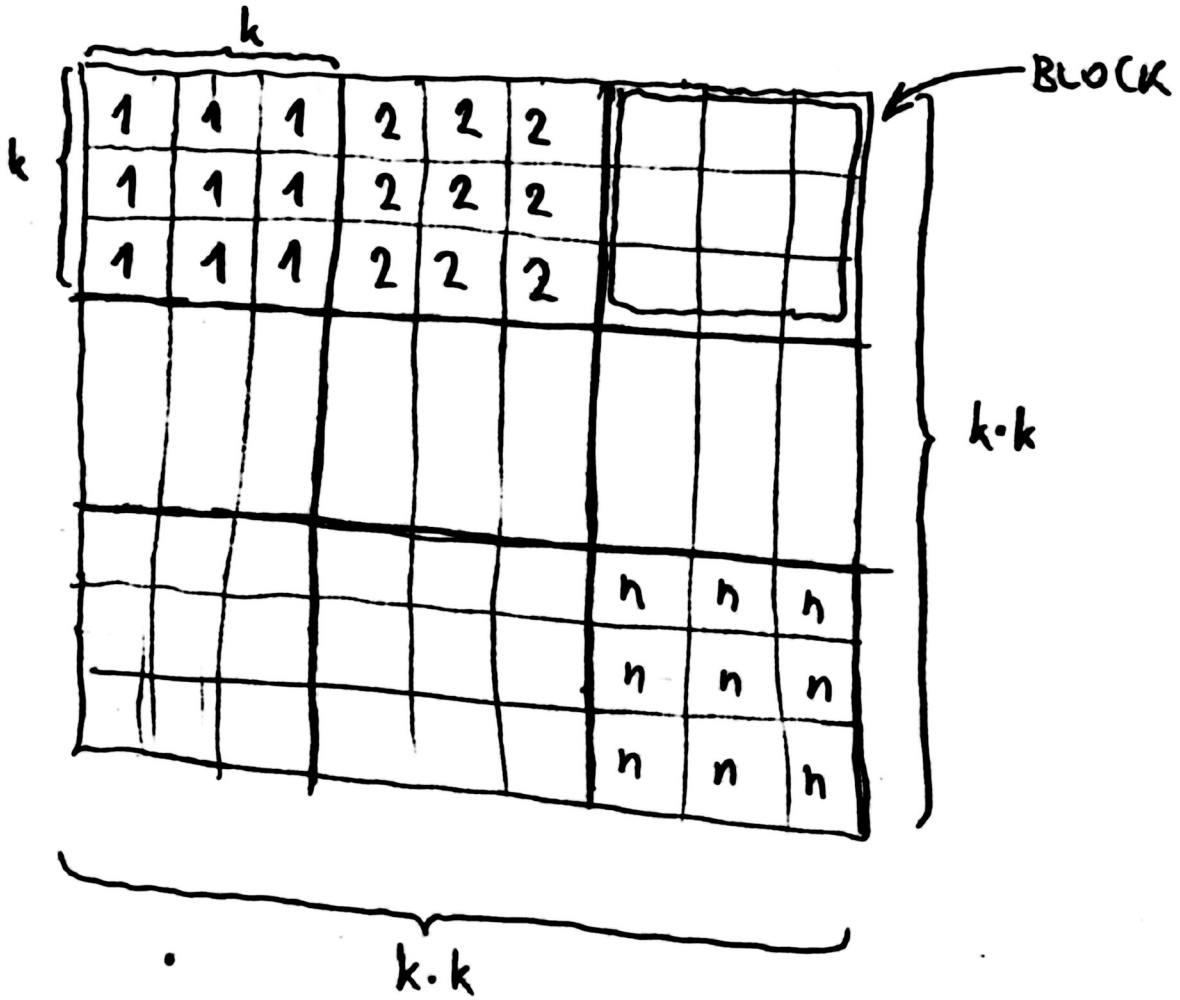
DM1012 HOMEWORK

PROBLEM 4:

b) FOR $n = k^2$ $k \in \mathbb{N}$ WE CAN CONSTRUCT $M \in [n]^{n \times n}$ SUCH AS IN THE PICTURE:



EACH ROW AND COLUMN CONTAINS EXACTLY $k = \sqrt{n}$ DIFFERENT ELEMENTS. ∎

WE HAVE $k \cdot k$ BLOCKS EACH WITH k^2 EQUAL ELEMENTS.

$\Rightarrow$ TOGETHER n^2 ELEMENTS IN THE MATRIX M.

THE BLOCKS HAVE NUMBERS $1, 2, \ldots, n = k \cdot k$

EACH BLOCK HOLDS ONLY ONE NUMBER.

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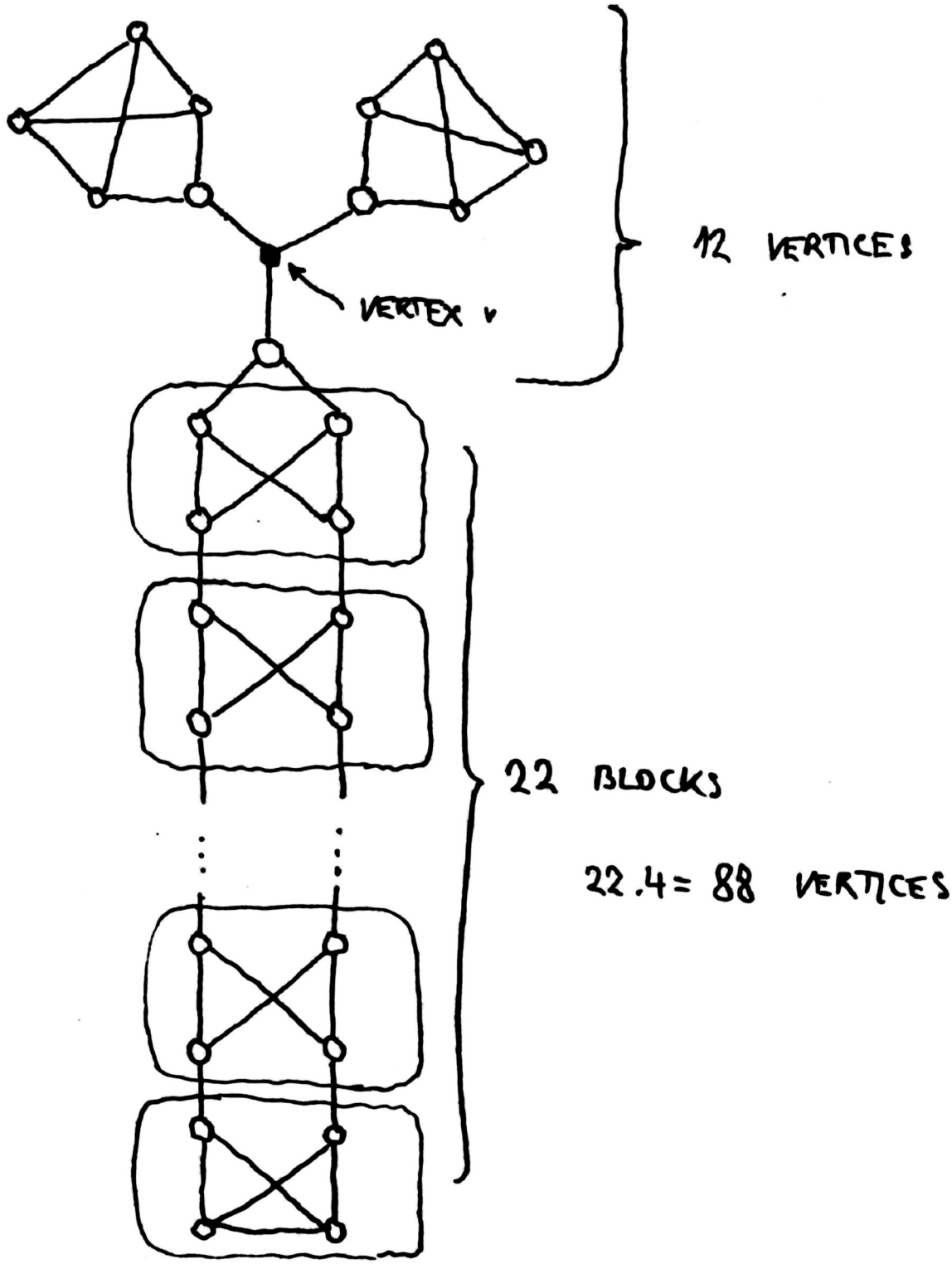
PROBLEM 5 :

G:



G IS A CONNECTED 3-REGULAR GRAPH WITH 100 VERTICES THAT DOES NOT SATISFY THE TUTTE CONDITION, BECAUSE IF WE REMOVE THE VERTEX v, WE GET 3 ODD COMPONENTS. ✓ 2/2

10.11.2007 Sa
PETER ČERNO

DM1012 HOMEWORK

PROBLEM 11

a) LET $x_1, x_2, \dots, x_{rs+1} \in \mathbb{R}$

LET US DEFINE $\prec \subseteq [rs+1]^2$ AS $i \prec j \overset{\text{def.}}{\Leftrightarrow} i < j \wedge x_i \leq x_j$ $\Rightarrow$ WE DEFINED A PARTIAL ORDER ON $[rs+1]$.

DILWORTH THEOREM SAYS: FOR EVERY FINITE PARTIALLY ORDERED SET X THE MINIMUM NUMBER OF COVERING CHAINS EQUALS TO THE LENGTH OF THE MAXIMUM ANTICHAIN.

LET $A_1, \dots, A_m$ ARE THE COVERING CHAINS WITH m MINIMAL.

AND LET B BE THE MAXIMAL ANTICHAIN, $|B| = m$.

1) IF $m \leq s$ THEN (A_i ARE PAIRWISE DISJOINT)

FROM $rs+1 = |A_1 \cup \dots \cup A_m| = |A_1| + \dots + |A_m|$

FOLLOWS THAT THERE IS $i \in [m]$ S.T. $|A_i| > r$.

(IF $\forall i \in [m]$ $|A_i| \leq r \Rightarrow \sum_{i=1}^{m} |A_i| \leq mr \leq rs < rs+1$ ↯)

IF WE TAKE A SUBSEQUENCE OF $x_1, \dots, x_{rs+1}$ WITH INDICES IN A_i, WE GET A NONDECREASING SUBSEQUENCE OF LENGTH AT LEAST $r+1$.

2) IF $m > s$ I.E. $|B| \geq s+1$ $\Rightarrow$ IF WE TAKE A SUBSEQUENCE OF $x_1, \dots, x_{rs+1}$ WITH INDICES IN B, WE GET A DECREASING SEQUENCE OF LENGTH AT LEAST $s+1$.

WHY? IF FOR $i, j \in B$ $i < j$ $x_i \leq x_j$, THEN i AND j ARE COMPARABLE. ↯

b) LET US ASSUME THAT $\forall i \in [rs+1]$

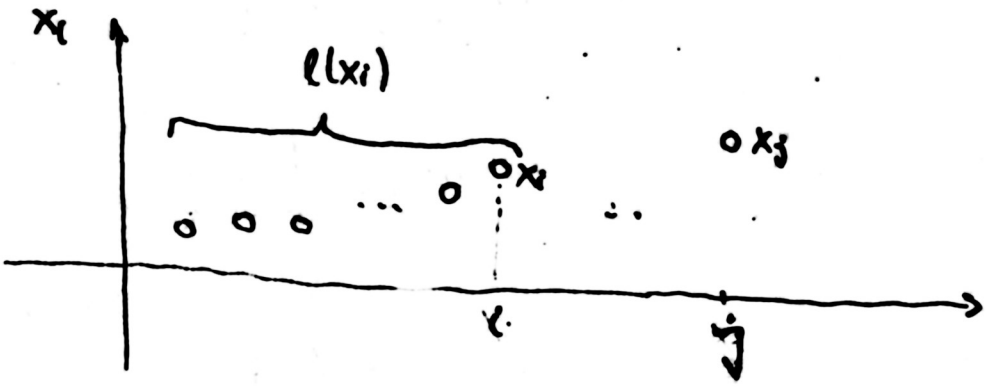
$1 \leq \ell(x_i) \leq r \wedge 1 \leq \ell'(x_i) \leq s$

IF WE LOOK AT ALL $rs+1$ PAIRS $(\ell(x_i), \ell'(x_i))$ THEN THERE EXIST $i, j \in [rs+1]$ $i \neq j$ SUCH THAT $(\ell(x_i), \ell'(x_i)) = (\ell(x_j), \ell'(x_j))$

WHY? BECAUSE FOR A PAIR (a, b) WE HAVE $r \cdot s$ POSSIBILITIES AND THERE ARE $rs+1$ PAIRS.

LET US ASSUME THAT $i < j$.

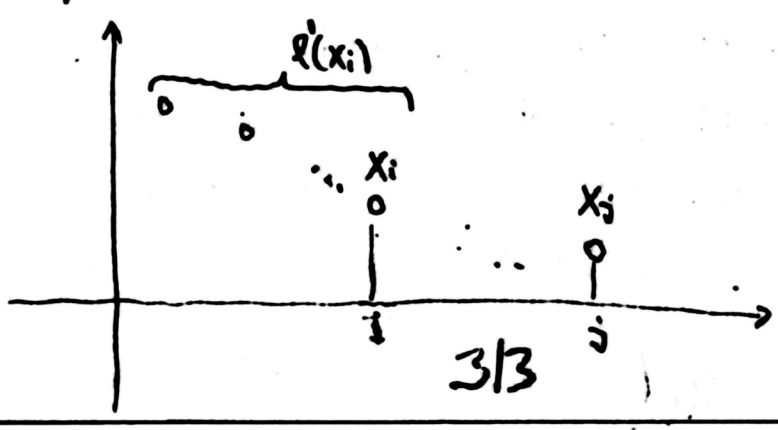
a) $x_i \leq x_j$



WE CAN SEE THAT WE CAN FIND A NONDECREASING SUBSEQUENCE OF LENGTH $\ell(x_i)+1$ ENDING IN x_j

$\Rightarrow$ CONTRADICTION WITH $\ell(x_i) = \ell(x_j)$. ↯

b) $x_i > x_j$



AGAIN WE CAN FIND A DECREASING SUBSEQUENCE OF LENGTH $\ell'(x_i)+1$ ENDING IN x_j,

CONTRADICTION WITH $\ell'(x_i) = \ell'(x_j)$. ↯